# Continuous monitoring and the introduction of a classical level in Quantum Theory

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**Abstract**

In ordinary Quantum Mechanics only ideally instantaneous observations of a quantity or a set of compatible quantities are usually considered. In an old paper of our group in Milano a formalism was introduced for the continuous monitoring of a system during a certain interval of time in the framework of a somewhat generalized approach to Q. M. The outcome was a distribution of probability on the space of all the possible continuous histories of a set of quantities to be considered as a kind of coarse grained approximation to some ordinary quantum observables commuting or not. The main aim was the introduction of a classical level in the context of Quantum Mechanics, treating formally a set of basic quantities *to be considered as beables* in the sense of Bell as continuously taken under observation. However the effect of such assumption was a permanent modification of the Liouville-von Neumann equation for the statistical operator by the introduction of a dissipative term which is in conflict with basic conservation rules in all reasonable models we had considered. Difficulties were even encountered for a relativistic extension of the formalism. In this paper I propose a modified version of the original formalism which seems to overcome both difficulties. First I study the simple models of an harmonic oscillator and a free scalar field in which a coarse grain position and a coarse grained field respectively are treated as beables. Then I consider the more realistic case of Spinor Electrodynamics in

which only certain coarse grained electric and magnetic fields and no matter related quantities are introduced as classical variables.

# 1 Introduction

In reference [1] a general formalism was introduced for the treatment of the *continuous monitoring* of a quantity or a set of quantities in the framework of Quantum Mechanics. The formalism turns out to be strictly related to a more particular one previously proposed by E. B. Davies for the observation of the counting times on a system of counters [2]. This is in the context of the generalized formulation of Quantum Mechanics (GQM) based on the concept of positive operator valued measures (p.o.m.) and operation valued measures or instruments, originally proposed by G. Ludwig, E. B. Davies, S. Holevo [2]-[4]. The outcome is a distribution of probability on the set of all the possible continuous histories of the monitored quantities in the considered time interval. The class of event subsets is characterized in terms of time average of the monitored quantities of the type

$$a_h^s(t) = \int dt' h(t-t')\, a^s(t')\,, \tag{1}$$

the $h(t)$s being appropriate weight functions (generalized stochastic process).

Later various alternative formulations of the theory have been given and various aspects developed with interesting applications, particularly in the field of Quantum Information and Optics (for a recent presentation see e. g. [5] and references therein).

The original purpose of [1] was, however, to obtain a modification of ordinary Quantum Theory, in which an intrinsic classical level for some basic macroscopic quantities could be introduced, to solve consistently the problem of interpretation of the Theory in the sense of Bohr and of Von Neumann. The idea was that certain basic quantities should be chosen once for ever as an additional postulate and they should be thought as having at any time a well defined value, considered *beables* in the sense of Bell, by treating them formally as continuously observed. Then any other observation on a microscopic subsystem should be expressed in terms of the modifications that its interaction with the remaining part of the world produces in the value

of such basic macroscopic quantities. Obviously the theory remain statistic and to any possible evolution of the basic quantities a precise probability is assigned. Note that, even if in a completely different mathematical framework, similar conceptual ideas seem to be at the basis of the so called theory of the *consistent histories* [6]. With reference to them see however also ref. [7].

A significant property of the modified theory is that integrating over all possible histories of the basic quantities in a given time interval $(t_0, t_F)$ is equivalent to introduce a dissipative term in the Liouville-Von Neumann evolution equation for the statistical operator in the Schroedinger picture. A particular interesting case of such a modified Liouville-Von Neumann equation is

$$\frac{\partial \hat{\rho}_S(t)}{\partial t} = -i[\hat{\rho}_S(t), \hat{H}] - \sum_j \alpha_j [\hat{A}^j, [\hat{A}^j, \hat{\rho}_S(t)]], \tag{2}$$

where $\hat{A}^1$, $\hat{A}^2$, ... denote hermitian operators not necessarily each other commuting and $\alpha_1, \alpha_2, \ldots$ positive constants. Alternatively in Heisenberg picture, which we find more convenient in this paper, this corresponds to ascribe a time dependence to the statistical operator according to the equation

$$\frac{\partial \hat{\rho}(t)}{\partial t} = -\sum_j \alpha_j [\hat{A}^j(t), [\hat{A}^j(t), \hat{\rho}(t)]]. \tag{3}$$

In the case the monitored quantities $a^1$, $a^2$, ... may be considered simultaneous coarse grained approximations of the ordinary quantum observables $A^1$, $A^2$, ... with $\langle a^s(t)\rangle = \langle \hat{A}^s(t)\rangle$ and the variance $\langle (a^s_h(t) - \langle a^s_h(t)\rangle)^2\rangle$ expressed as the sum of an intrinsic term independent of $\hat{\rho}$ and a minor modification of the ordinary quantum variance for $A^s(t)$. To $a^1$, $a^2$, ... in the following we shall conventionally refer as the macroscopic $A^1$, $A^2$, ...

Notice that eqs. (2) or (3) are trace and positivity preserving. The dissipative term expresses the permanent effect of the modification introduced in the theory (formally the perturbation produced by the continuous monitoring), even when any information on the mentioned basic quantities is completely disregarded. The above equations make also our theory in contact with theories that introduce ad hoc dissipative terms, as a noise, to simulate the interaction of the apparatus with an environment (see in particular in this connection ref. [8]) or theories that want introduce an intrinsic progressive decoherence and a spontaneous collapse at a more fundamental

level. Theories of the latter type received considerable attention in the last thirty years (see e. g. [9] for general revues and complete references; a small representative sample, corresponding to various point of view, is reported in refs. [10]-[13]).

In our context specific examples of choice of the basic macroscopic quantities in simple models may be the macroscopic density of particle in a non-relativistic second quantization theory with self-interaction, the particle distribution functions in the classical phase space for a system of electrons and protons [14], a macroscopic field for a self-interacting quantum scalar field [1]. Unfortunately in all examples we have considered eqs. (2) or (3) turn out to be in conflict with important conservation rules (the energy conservation rule in particular) at a level that does not seem compatible with the matter stability as presently established [1], [14] (in this connection see however even [10]). There are also difficulties to a covariant extension of the formalism to a vector field, like the electro-magnetic field, or to a tensor one and, even in the case of the scalar field, the additional term in (2, 3) would be ill defined.

All difficulties seem to be related to the requirement that $\alpha_1, \alpha_2, \ldots$ were positive. In this paper we want to show that it is possible to release such requirement and significant models can be constructed in which by an appropriate choice of the operators $\hat{A}^1, \hat{A}^2, \ldots$, of the constants $\alpha_1, \alpha_2, \ldots$ and of the weight functions $h(t)$ a consistent probability distribution for the histories of related basic quantities can be defined and the conservation rules respected. We have to assume that $\hat{\rho}$ is trace 1 and positive at some initial time $t_0$ but it is not necessary that the positivity of $\hat{\rho}(t)$ is preserved in time to have a positive probability for the quantities of interest.

Among the above models, it is of particular interest the case of spinor QED in which the basic quantities to be considered as *classical* or *beables* are the macroscopic components of the electromagnetic field, but nothing concerning matter. In this case eq. (3) becomes

$$\frac{\delta \hat{\rho}[\sigma]}{\delta \sigma(x)} = \frac{\gamma}{16} \left[ \hat{F}_{\mu\nu}(x), \left[ \hat{F}^{\mu\nu}(x), \hat{\rho}[\sigma] \right] \right] , \tag{4}$$

where $\sigma$ is a space-like surface passing across $x$. Note that the form of this equation is practically completely determined by Lorentz and gauge invariance requirements and that it corresponds to $\alpha_1$, $\alpha_2$, ... not all positive. Furthermore as in any field theory the averages (1) in terms of which the

probabilities are defined have to be replaced by expressions of the type

$$f_h^{\mu\nu}(x) = \int d^4x \, h(x-x') \, f^{\mu\nu}(x') \tag{5}$$

and the restriction on the $h(x)$s consists in the requirement that only time like wave vectors $k$ occur in the Fourier transform $\tilde{h}(k)$ or that they are dominant.

The plan of the paper is the following one. In section 2 we shall review the formalism of the continuous monitoring, manly to establish notations, and recall the important notion of functional generator. In sect 3 we shall consider the case of the harmonic oscillator in the original formulation, assuming as basic quantity the macroscopic position and show the positivity of the corresponding probability by using path integral techniques; then we extend discussion to the case of a free scalar field. In sect. 4 we discuss the problem of the conservation rules and in sect. 5 we show how the formalism can be consistently modified in order to dispose of the corresponding violation always for the harmonic oscillator and the scalar field. In sect. 6 we consider the mentioned more significant case of spinorial QED. Finally in sect. 7 we summarized the results and try to make some conclusions and additional remarks. Some technicalities are confined in the appendices.

# 2 Continuous monitoring of a set of quantities

In GQM a set of *compatible observables* $A \equiv (A^1, A^2, \ldots A^p)$ is associated to a normalized *effect* or *positive operator valued measure* (p.o.m.) $\hat{F}_A(T)$ and the apparatus $S_A$ for observing them to an *instrument* or *operation valued measure* (o.v.m.) $\mathcal{F}_{S_A}(T)$, $T$ being a Borel subset of the real space $\Re^p$ of all possible values of $A$.

That is, $\hat{F}_A(T)$ and $\mathcal{F}_{S_A}(T)$ are a positive operator on the Hilbert space $\mathcal{H}$ associated to the system and a mapping of the set of the trace class operators in itself, respectively, satisfying the relations

$$\hat{F}(\cup_{j=1}^n T_j) = \sum_{j=1^n} \hat{F}(T_j) \qquad \text{and} \qquad \mathcal{F}(\cup_{j=1}^n T_j) = \sum_{j=1^n} \mathcal{F}(T_j) \,, \tag{6}$$

if

$$T_i \cap T_j = 0 \tag{7}$$

and

$$\hat{F}_A(\Re^p) = \hat{I} \qquad \mathrm{Tr}[\mathcal{F}_{S_A}(\Re^p)\hat{X}] = \mathrm{Tr}\hat{X}\,. \tag{8}$$

Further they must be related each other by the equation

$$\hat{F}_A(T) = \mathcal{F}'_{S_A}(T)\hat{I}\,, \tag{9}$$

where by $\mathcal{F}'$ we denote the dual mapping of $\mathcal{F}$, defined by the equation

$$\mathrm{Tr}\left[\hat{B}\mathcal{F}\hat{X}\right] = \mathrm{Tr}\left[\left(\mathcal{F}'\hat{B}\right)\hat{X}\right]\,, \tag{10}$$

$\hat{X}$ being an arbitrary trace class operator and $\hat{B}$ an arbitrary bounded operator. So

$$\mathrm{Tr}[\hat{F}_A(T)\hat{X}] = \mathrm{Tr}[\mathcal{F}_{S_A}(T)\hat{X}]\,. \tag{11}$$

As we told, we shall find convenient to work in Heisenberg picture. Then we have

$$\begin{gathered}\hat{F}_A(T,t) = e^{iHt}\hat{F}_A(T)e^{-iHt} \\ \mathcal{F}_{S_A}(T,t)\hat{X} = e^{iHt}[\mathcal{F}_{S_A}(T)(e^{-iHt}\hat{X}e^{iHt})]e^{-iHt}\end{gathered} \tag{12}$$

and the probability of observing $A \in T$ at the time $t$ is

$$P(A \in T, t|\rho) = \mathrm{Tr}[\hat{F}_A(T,t)\hat{\rho}] = \mathrm{Tr}[\mathcal{F}_{S_A}(T,t)\hat{\rho}]\,, \tag{13}$$

where $\hat{\rho}$ denotes the statistical operator representing the state of the system (a priori a mixture state), which we usually indicate by $\rho$.

The reduction of the state as consequence of having observed $A \in T$ at the time $t_0$ by the apparatus $S_A$ must be written as

$$\hat{\rho} \to \mathcal{F}_{S_A}(T,t_0)\hat{\rho}/\mathrm{Tr}[\mathcal{F}_{S_A}(T,t_0)\hat{\rho}]\,. \tag{14}$$

Notice

$$\langle A^j \rangle = \mathrm{Tr}[\hat{A}^j(t)\hat{\rho}] \tag{15}$$

with

$$\hat{A}^j(t) = e^{iHt}\hat{A}^j e^{-iHt} \qquad \text{and} \qquad \hat{A}^j = \int_{\Re^p} d\hat{F}(a)a^j\,. \tag{16}$$

The operators $\hat{A}^j$ are Hermitian but generally they do not commute. Such a set of generalized compatible observables can be interpreted as corresponding to an approximate simultaneous measurement of possibly incompatible ordinary observables $\hat{A}_1$, $\hat{A}_2$, ...

Now let us assume that we make repeated *independent* observations on $A$ at subsequent times $t_0, t_1, \ldots t_N$. Combining eqs. (13) and (14) the *Joint probability* of observing a *a sequence of results* for $A$ can be written

$$\begin{aligned} & P(A \in T_N, t_N; \ldots A \in T_1, t_1; A \in T_0, t_0|\rho) = \\ & \quad = \mathrm{Tr}[\mathcal{F}_{S_A}(T_N, t_N) \ldots \mathcal{F}_{S_A}(T_1, t_1)\mathcal{F}_{S_A}(T_0, t_0)\hat{\rho}] \end{aligned} \tag{17}$$

Notice that

$$\mathcal{F}(T_N, t_N; \ldots; T_1, t_1; T_0, t_0) = \mathcal{F}_{S_A}(T_N, t_N) \ldots \mathcal{F}_{S_A}(T_1, t_1)\mathcal{F}_{S_A}(T_0, t_0)) \tag{18}$$

and

$$\hat{F}(T_N, t_N; \ldots; T_1, t_1; T_0, t_0) = \mathcal{F}'_{S_A}(T_0, t_0)\mathcal{F}'_{S_A}(T_1, t_1) \ldots \mathcal{F}'_{S_A}(T_N, t_N))\hat{I} \tag{19}$$

define an instrument and a p.o.m. on a real space with $p(N+1)$ dimensions $\Re^{p(N+1)}$.

Then

$$\begin{aligned} P(A \in T_N, t_N; \ldots; A \in T_0, t_0|\rho) = & \\ = \mathrm{Tr}[\mathcal{F}(T_N, t_N; \ldots; T_0, t_0)\hat{\rho}] & \\ = \mathrm{Tr}[\hat{F}(T_N, t_N; \ldots; T_0, t_0)\hat{\rho}]\,. & \end{aligned} \tag{20}$$

So in GQM the observation of a sequence of results at certain successive times can be put on the same foot as the observation of $A$ at a single time.

## 2.1 Continuous monitoring

On analogy with above let us consider the limit case of a set of quantities continuously kept under observation. [1]

[1]Note that in the framework of ordinary quantum mechanics, in which only exact observations are considered corresponding to projection valued measures, there is a negative theorem in connection with this problem, usually recalled as Zeno's theorem. According

Let $\mathcal{Y}$ be the functional space of all possible *histories* $a(t) \equiv (a^1(t), a^2(t), \ldots a^p(t))$ of a set of quantities in a reference time interval $(t_0, t_F)$ (where $t_F$ may be possible taken to $+\infty$) and $\Sigma$ the class of the measurable subsets of $\mathcal{Y}$ according to some definition to be specified later. Furthermore let us denote by $\Sigma_{t_a}^{t_b} \subset \Sigma$ the class of the measurable subsets of $\mathcal{Y}$ corresponding to restrictions on the histories only in the interval $(t_a, t_b) \subset (t_0, t_F)$.

Then we assume that an *instrument* and a related *p.o.m.*

$$\mathcal{F}(t_b, t_a; M) \qquad \text{and} \qquad \hat{F}(t_b, t_a; M) = \mathcal{F}'(t_b, t_a; M)\hat{I} \tag{21}$$

are defined, with $M \in \Sigma_{t_a}^{t_b}$ and we interpret

$$P(t_b, t_a; M) = \mathrm{Tr}[\hat{F}(t_b, t_a; M)\hat{\rho}(t_a)] = \mathrm{Tr}[\mathcal{F}(t_b, t_a; M)\hat{\rho}(t_a)] \tag{22}$$

as the *probability of observing* $a(t) \in M$ during the interval of time $(t_a, t_b)$ $\hat{\rho}(t_a)$ being the statistical operator at the time $t_a$ (see later). We shall call $\hat{F}(t_b, t_a; M)$ and $\mathcal{F}(t_b, t_a; M)$ a positive operator valued stochastic process and an operation valued stochastic process (OVSP), respectively.

We assume that $\mathcal{F}(t_b, t_a; M)$ satisfies the relation

$$\mathcal{F}(t_c, t_b; N)\mathcal{F}(t_b, t_a; M) = \mathcal{F}(t_c, t_a; N \cap M) \tag{23}$$

which as above it expresses the independence of the observation of $a(t)$ in successive intervals of time. Notice that

$$M \in \Sigma_{t_a}^{t_b}\,, \quad N \in \Sigma_{t_b}^{t_c} M \quad \Rightarrow \quad N \cap M \in \Sigma_{t_a}^{t_c}\,. \tag{24}$$

Under the above assumptions,

$$P(t_c,\, t_a;\, N \cap M) = \mathrm{Tr}[\hat{F}(t_c, t_b; N)\mathcal{F}(t_b, t_a; M)\hat{\rho}(t_b)] \tag{25}$$

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to such theorem, if we make repeated observations of the same quantity and let go to 0 the interval of time $\tau$ between two subsequent observations, the value of the quantity is frozen to its initial value and does not longer change with time. However, in GQM is possible to consider a double limit in which, as $\tau \to 0$, the observation in itself is made progressively less precise in such way that a finite result is attained. The discussion in this section can be thought in this perspective. The result is a generalized stochastic process in the sense of Gelfand, in which a set of histories of finite measure of the quantities of interest has to be specified in terms successive time averages of the type (1) of the quantities of interest rather then of values assumed at definite times (see subsec. 2.3).

is the *joint probability* of observing $a(t) \in M$ in the time interval $(t_a, t_b)$ and $a(t) \in N$ in the interval $(t_b, t_c)$. Then let us introduce the mapping

$$\mathcal{G}(t_b, t_a) \equiv \mathcal{F}(t_b, t_a; \mathcal{Y}) \tag{26}$$

and notice that for the normalization requirement it must be trace preserving.

By setting $M = \mathcal{Y}$ in (25) we find

$$P(t_c,\, t_b;\, N) = P(t_C,\, T_A;\, N \cup \mathcal{Y}) = \mathrm{Tr}[\hat{F}(t_c, t_b; N)\mathcal{G}(t_b, t_a)\hat{\rho}]. \tag{27}$$

Then for comparison with (22) we can set $\hat{\rho}(t_b) = \mathcal{G}(t_b, t_a)\hat{\rho}(t_a)$ and so the *perturbation* produced on the system by its continuous observation introduces a kind of time dependence on the statistical operator in Heisenberg picture that is described by the action of the operator $\mathcal{G}(t_b, t_a)$.

As we mentioned in the introduction we shall assume that our basic classical quantities are formally treated as continuously observed. In this perspective the OVSP $\mathcal{F}(t_b, t_a; M)$ must be chosen once for all and considered a part of the theory and the action of the related $\mathcal{G}(t_b, t_a)$ expresses the modification introduced in this way in the dynamics of the ordinary theory.

## 2.2 Characteristic functional operator

On analogy with the usual probability theory, we define the functional Fourier transform or *characteristic functional operator* (CFO)

$$\mathcal{G}(t_b, t_a; [\xi(t)]) = \int \mathcal{F}(t_b, t_a; \mathcal{D}_c M) \exp\left\{ -i \int_{t_a}^{t_b} dt \xi^s(t) a^s(t) \right\} , \tag{28}$$

where $\mathcal{D}_c M$ denotes the measure of an elementary set in the functional space $\mathcal{Y}$ (the index “c” refers to the interpretation of $a(t)$ as a classical history, to distinguish the classical functional measure from the quantum path integral measure which shall be used in the following). The concept of CFO turns out to be very useful not only to study the properties of a given OVSP but even to construct such a structure.

Notice that in terms of $\mathcal{G}(t_b, t_a; [\xi(t)])$ assumption (23) becomes

$$\mathcal{G}(t_c, t_b; [\xi(t)])\, \mathcal{G}(t_b, t_a; [\xi(t)]) = \mathcal{G}(t_c, t_a; [\xi(t)])\,. \tag{29}$$

Then, if we set

$$\mathcal{G}(t+dt,t;[\xi(t)]) = 1 + \mathcal{K}(t;\xi(t))dt\,, \tag{30}$$

we can write the differential equation

$$\frac{\partial}{\partial t}\,\mathcal{G}(t,t_a;[\xi]) = \mathcal{K}(t;\xi(t))\,\mathcal{G}(t,t_a;[\xi])\,, \tag{31}$$

that we can formally solve as

$$\mathcal{G}(t_b,t_a;[\xi]) = \mathrm{T}\exp\int_{t_a}^{t_b} dt\mathcal{K}(t;\xi(t))\,, \tag{32}$$

T being the usual time ordering prescription.

Now let us observe that

$$\mathcal{G}(t_b,t_a;[0]) = \mathcal{F}(t_b,t_a;\mathcal{Y}) \equiv \mathcal{G}(t_b,t_a). \tag{33}$$

Then setting $\xi(t)=0$ in (31) we obtain also

$$\frac{\partial}{\partial t}\,\mathcal{G}(t,t_a) = \mathcal{L}(t)\,\mathcal{G}(t,t_a) \tag{34}$$

and

$$\mathcal{G}(t_b,t_a) = \mathrm{T}\exp\int_{t_a}^{t_b} dt\mathcal{L}(t)\,, \tag{35}$$

with

$$\mathcal{L}(t) = \mathcal{K}(t;0)\,. \tag{36}$$

Furthermore, being $\mathcal{G}(t_b,t_a)$ trace preserving, we must have

$$\mathrm{Tr}\{\mathcal{L}(t)\hat{\rho}\} = 0\,. \tag{37}$$

Under some additional assumption, eq. (37) and the requirement $\mathcal{F}(t_b,t_a;M)$ and so $\mathcal{G}(t_b,t_a)$ be a *positive mapping* (actually a *completely* positive mapping) imply $\mathcal{L}(t)$ to be of the general form

$$\mathcal{L}(t)\hat{\rho} = -\sum_{s=1}^{p}\alpha_s(\hat{R}^{s\dagger}\hat{R}^s\hat{\rho} + \hat{\rho}\hat{R}^{s^\dagger}\hat{R}^s - 2\hat{R}^s\hat{\rho}\hat{R}^{s\dagger})\,. \tag{38}$$

$\alpha_s$ being appropriate positive constants (cf. [15]).

Conversely we can set

$$\mathcal{F}(t_b, t_a; \mathcal{D}_c M) = \mathbf{f}(t_b, t_a; [a(t)])\, \mathcal{D}_c M\,, \tag{39}$$

with

$$\begin{aligned} &\mathbf{f}(t_b, t_a; [a(t)]) = \\ &= \int \mathcal{D}_c\xi \,\exp\left\{ i \sum_{s=1}^{p} \int_{t_a}^{t_b} dt\, \xi^s(t)\, a^s(t) \right\} \mathcal{G}(t_b, t_a; [\xi(t)]\,, \end{aligned} \tag{40}$$

where the measure $\mathcal{D}_c\xi$ is normalized in such a way that

$$\int \mathcal{D}_c\xi \,\exp\left\{ -i \sum_{s=1}^{p} \int_{t_a}^{t_b} dt\, \xi^s(t)\, (a^s(t) - a'^s(t)) \right\} = \delta([a(t)] - [a'(t)])\,, \tag{41}$$

$\delta([a(t)] - [a'(t)])$ being the $\delta$ - functional with respect to the measure $\mathcal{D}_c a$.

Formally, this may be achieved assuming the interval $(t_a, t_b)$ divided in $N$ equal parts of amplitude $\epsilon = (t_b - t_a)/N$, define

$$\mathcal{D}_c M \equiv \mathcal{D}_c a = \left(\frac{\epsilon}{2\pi}\right)^{Np/2} d^p a_1 \ldots d^p a_N\,, \qquad \mathcal{D}_c \xi = \left(\frac{\epsilon}{2\pi}\right)^{Np/2} d^p \xi_1 \ldots d^p \xi_N\,, \tag{42}$$

and

$$\delta\left([a(t)] - [a'(t)]\right) = \left(\frac{2\pi}{\epsilon}\right)^{Np/2} \delta^p(a_1 - a_2') \ldots \delta(a_N - a_n'), \tag{43}$$

to be understood in the limit $N \to \infty$.

Eqs. (39) and (40) enable us to reconstruct $\mathcal{F}(t_b, t_a; M)$ given $\mathcal{G}(t_b, t_a; [\xi])$ or $\mathcal{K}(t, \xi(t))$. Naturally, $\mathcal{K}(t, \xi(t))$ has to be of an appropriate form in order that $\mathbf{f}(t_1, t_0; [a(t)])$, as defined by (40) be completely positive.

Two such forms are known: the *Gaussian form*, the *Poissonian form* and obviously a sum of the two.

*Gaussian form*:

$$\mathcal{K}(t, \xi(t))\hat{\rho} = \mathcal{L}(t)\hat{\rho} - i \sum_{s=1}^{p} \xi^s(t)(\hat{R}^s(t)\hat{\rho} + \hat{\rho}\hat{R}^{s\dagger}(t)) - \sum_{s=1}^{p} \frac{1}{4\alpha_s} \xi^{s\,2}(t)\,\hat{\rho} \tag{44}$$

*Poissonian form*:

$$\mathcal{K}(t,\xi(t))\hat{\rho} = \mathcal{L}(t)\hat{\rho} + 2\sum_{s=1}^{p}\alpha_s(e^{-i\xi^s(t)/2\alpha_s} - 1)(\hat{R}^s(t)\hat{\rho}\hat{R}^{s\dagger}(t))\,. \tag{45}$$

Notice that from (40), (41) we have

$$\int \mathcal{D}_{\rm c}a\,\mathbf{f}(t_b,t_a;[a(t)]) = \int \mathcal{D}_{\rm c}\xi\;\delta([\xi])\,\mathcal{G}(t_b,t_a;[\xi]) = \mathcal{G}(t_b,t_a) \tag{46}$$

and so

$$\int \mathcal{D}_{\rm c}a\,\mathrm{Tr}\,\{\mathbf{f}(t_b,t_a;[a(t)])\hat{\rho}(t_a)\} = \mathrm{Tr}\hat{\rho}(t_a) = 1\,. \tag{47}$$

Likewise, for the momenta of the components of $a(t)$ at certain definite times $t_1, t_2, \ldots t_N$ in the interval $(t_a, t_b)$

$$\langle a^{s_1}(t_1)\,a^{s_2}(t_2)\ldots a^{s_l}(t_l)\rangle = \int D_{\rm c}a\;a^{s_1}(t_1)\,a^{s_2}(t_2)\ldots a^{s_l}(t_l) \tag{48}$$
$$\mathrm{Tr}\,\{\mathbf{f}(t_b,t_a;[a(t)])\hat{\rho}(t_a)\} = i^l\,\mathrm{Tr}\left\{\frac{\delta}{\delta\xi^{s_1}(t_1)}\cdots\frac{\delta}{\delta\xi^{s_l}(t_l)}\mathcal{G}(t_b,t_a;[\xi])\hat{\rho}(t_a)\right\}\Big|_{\xi=0}.$$

In particular we have for the expectation value of a single component

$$\langle a^s(t)\rangle = \mathrm{Tr}\left[i\,\frac{\delta}{\delta\xi^s(t)}\;\mathcal{G}(t_b,\,t_a;\,[\xi])|_{\xi=0}\;\hat{\rho}(t_a)\right] = \tag{49}$$
$$= \mathrm{Tr}\left[\mathcal{G}(t_b,\,t)\,i\,\frac{\partial}{\partial\xi^s}\,\mathcal{K}(t,\xi)|_{\xi=0}\,\mathcal{G}(t,t_a)\,\hat{\rho}(t_a)\right] = \mathrm{Tr}[\hat{A}^s\,\mathcal{G}(t,t_a)\,\hat{\rho}(t_a)\,]$$

with

$$\hat{A}^s(t) = \hat{R}^s(t) + \hat{R}^{s\dagger}(t) \tag{50}$$

in the *Gaussian* case and

$$\hat{A}^s(t) = \hat{R}^{s\dagger}(t)\hat{R}_s(t) \tag{51}$$

in the *Poissonian* case.

Therefore we can talk of $a^1(t), a^2(t), \ldots, a^p(t)$ as the value at time $t$ of the macroscopic or, in our interpretation, the classical counterpart of the quantum observable associated to $\hat{A}^s$ even if $A^1, A^2, \ldots, A^p$ do not each others commute.

For the second momenta we have

$$\langle a^s(t)\, a^{s'}(t')\rangle = \delta(t-t')\,\mathrm{Tr}\left\{\frac{\partial^2\mathcal{K}(t,\xi)}{\partial\xi^s\partial\xi^{s'}}\mathcal{G}(t,t_a)\hat\rho\right\}\Big|_{\xi=0} -$$
$$-\theta(t-t')\,\mathrm{Tr}\left\{\frac{\partial K(t,\xi)}{\partial\xi^s}\mathcal{G}(t,t')\frac{\partial K(t',\xi)}{\partial\xi^{s'}}\mathcal{G}(t',t_a)\hat\rho\right\}\Big|_{\xi=0} -$$
$$-\theta(t'-t)\,\mathrm{Tr}\left\{\frac{\partial K(t',\xi)}{\partial\xi^{s'}}\mathcal{G}(t',t)\frac{\partial K(t,\xi)}{\partial\xi^s}\mathcal{G}(t,t_a)\hat\rho\right\}\Big|_{\xi=0}. \qquad (52)$$

The occurrence of the $\delta$ term in eq. (52) shows that only *time averages* of the type

$$a_h = \int dt\, h(t)\cdot a(t) \equiv \int dt \sum_{s=1}^{p} h^s(t) a^s(t)\,, \qquad (53)$$

are significant, where the weight functions $h(t) \equiv (h^1(t),\ldots,h^p(t))$are elements of the dual space $\mathcal{Y}'$. Therefore the class $\Sigma$ of the measurable set in $\mathcal{Y}$ should be defined in terms of such quantities (cf. the following subsection). More simply, we may refer to density of probability of the form

$$p(a_1,\, h_1;\, a_2,\, h_2;\, \ldots a_l,\, h_l) = \qquad (54)$$
$$= \int \mathcal{D}_{\mathrm{c}}a\, \delta(a_1-\bar a_{h_1})\ldots\delta(a_l-\bar a_{h_l})\,\mathrm{Tr}\{\mathbf{f}(t_b,t_a;[a])\,\hat\rho\} = \frac{1}{(2\pi)^l}\int dk_1\ldots$$
$$\ldots dk_l\, e^{i(k_1a_1+\ldots k_la_l)}\int \mathcal{D}_{\mathrm{c}}a\, e^{-i(k_1\bar a_{h_1}+\ldots k_l\bar a_{h_l})}\,\mathrm{Tr}\{\mathbf{f}(t_b,t_a;[a])\,\hat\rho\} =$$
$$= \frac{1}{(2\pi)^l}\int dk_1\ldots dk_l\, e^{i(k_1a_1+n\ldots k_la_l)}\mathrm{Tr}\{\mathcal{G}(t_b,t_a;[k_1h_1+\ldots\, k_lh_l]\,\hat\rho\}\ ,$$

where $h_1(t), h_2(t),\ldots\, h_l(t)$ are independent elements of $\mathcal{Y}$ with support in $(t_a,t_b)$. A sensible choice could be

$$h_j(t) = \left(n_j^1\, h(t-t_j),\, n_j^2\, h(t-t_j),\,\ldots\, n_j^p\, h(t-t_j)\right)\ , \qquad (55)$$

$h(t)$ being a function different from zero only in a narrow neighbouring of $t=0$ such that $\int dt\, h(t) = 1$ ; $n_j$ for $j=1,2,\ldots l$ unitary vectors in the euclidean $p$ dimensional space $\mathbf{R}^p$; $t_1,t_2,\ldots t_l$ certain intermediate times between $t_a$ and $t_b$.

## 2.3 Characterization of the history space $\mathcal{Y}$ and of the measurable set class $\Sigma$

Typically one can choose $\mathcal{Y} = \mathcal{E}' \times \mathcal{E}' \times \ldots \mathcal{E}'$ and $\mathcal{Y}' = \mathcal{E} \times \mathcal{E} \times \ldots \mathcal{E}$, where $\mathcal{E}$ is the class of the function with compact support in the $t$-axis and infinitely differentiable everywhere with the possible exception of finite discontinuities on the border of the support and $\mathcal{E}'$ the dual space of $\mathcal{E}$ (a subset of the Schwartz distribution space; punctual spectrum is not allowed). The possibility of finite discontinuities on the border has to be admitted to give full meaning to eq.(29).

Furthermore we recall that, if $h_1(t), h_2(t), \ldots h_l(t)$ are a set of elements of $\mathcal{Y}'$ as above and $B$ a Borel set in $\mathbf{R}^n$, the subset of $\mathcal{Y}$

$$C(h_1, h_2, \ldots h_n; B) = \{a(t) \in \mathcal{Y}; (\bar{a}_{h_1}, \bar{a}_{h_1} \ldots \bar{a}_{h_1}) \in B\} \tag{56}$$

is called a *cylinder set*. Then $\Sigma$ can be identified with the $\sigma$-algebra generated by all the cylinder sets for any choice of $n$, $B \subset \mathbf{R}^n$ and of $h_1, h_2, \ldots h_n$. The sub-algebra $\Sigma_{t_a}^{t_b}$ is the same, but with $h_j$ with support in the interval $(t_a, t_b)$..

# 3 Two specific Gaussian examples

Let us now consider two specific simple examples, the case of a non relativistic harmonic oscillator and the case of a relativistic scalar field in which the quantities continuously monitored are a *macroscopic position* $q(t)$ and a *macroscopic field* $\varphi(x)$ respectively in the sense of eq. (50).

## 3.1 Harmonic oscillator

Let us write the Lagrangian

$$L = \frac{1}{2}(\dot{Q}^2 - \omega^2 Q^2)\,, \tag{57}$$

the quantum Hamiltonian

$$\hat{H} = \frac{1}{2}\,(\hat{P}^2 + \omega^2 \hat{Q}^2)\,, \tag{58}$$

and commutation rules

$$[\hat{Q}(t), \hat{P}(t)] = i\,. \tag{59}$$

Then we assume in (38, 44)

$$\hat{R}(t) = \hat{R}^\dagger(t) = \frac{1}{2}\hat{Q}(t) \tag{60}$$

and consequently

$$\mathcal{L}(t)\hat{\rho} = -\frac{\alpha}{4}[\hat{Q}, [\hat{Q}, \hat{\rho}]] \tag{61}$$

and

$$\mathcal{K}(t;\xi(t))\hat{\rho} = -\frac{\alpha}{4}[\hat{Q}, [\hat{Q}, \hat{\rho}]] - \frac{i}{2}\xi(t)\{\hat{Q}, \hat{\rho}\} - \frac{1}{4\alpha}\xi^2(t)\,, \tag{62}$$

$\alpha$ being a positive constant.

In view of the following developments we want reproduce a proof of the positivity of the corresponding $\mathbf{f}(t_b, t_a; [q(t)])$ as defined according to (40) using the formalism of the path integral [1]. Actually, in the perspective of our interpretation, we shall refer to the entire interval $(t_F,\, t_0)$, the restriction to a sub-interval being then trivial.

Let us divide the interval $(t_F, t_0)$ in $N$ infinitesimal parts with $\epsilon = (t_F - t_0)/N$ and $t_j = t_0 + j\epsilon$. Let us also set to simplify notation

$$\hat{\rho}_{[\xi]}(t) = \mathcal{G}(t, t_0; [\xi])\rho_0\,. \tag{63}$$

We have

$$\begin{aligned}\hat{\rho}_{[\xi]}(t_{j+1}) = \hat{\rho}_{[\xi]}(t_j) + \epsilon \Big\{ &-\frac{\alpha}{4}[\hat{Q}(t_j), [\hat{Q}(t_j), \hat{\rho}_{[\xi]}(t_j)]] \\ &-\frac{i}{2}\xi(t_j)\left(\hat{Q}(t_j)\rho_{[\xi]}(t_j) + \rho_{[\xi]}(t_j)\hat{Q}(t_j)\right) - \frac{\xi_j^2}{4\alpha}\hat{\rho}_{[\xi]}(t_j)\Big\}\end{aligned} \tag{64}$$

and, denoting by $|Q, t\rangle$ the eigenstates of $\hat{Q}(t)$,

$$\begin{aligned}\langle Q_{j+1}, t_{j+1}|\hat{\rho}_{[\xi]}(t_{j+1})|Q'_{j+1}, t_{j+1}\rangle &== \int dQ_j \int dQ'_j \\ &\langle Q_{j+1}, t_j + \epsilon|Q_j, t_j\rangle\langle Q_j, t_j|\hat{\rho}_{[\xi]}(t_j)|Q'_j, t_j\rangle\langle Q'_j, t_j|Q'_{j+1}, t_j + \epsilon\rangle \\ &\quad\left\{1 + \epsilon\left(-\frac{\alpha}{4}(Q_j - Q'_j)^2 - \frac{i}{2}\xi_j(Q_j + Q'_j) - \frac{\xi_j^2}{4\alpha}\right)\right\} =\end{aligned}$$

$$= \int \frac{dQ_j\, dP_j}{2\pi} \int \frac{dQ'_j\, dP'_j}{2\pi} \exp\left\{\left(iP_j(Q_{j+1}-Q_j) - \frac{\epsilon}{2}(P_j^2+\omega^2 Q_j^2)\right)\right\} \cdot$$
$$\langle Q_j, t_j|\hat\rho_{[\xi]}(t_j)|Q'_j, t_j\rangle \,\exp\left\{-\epsilon\left(\frac{\alpha}{4}(Q_j-Q'_j)^2 + \frac{i}{2}\xi_j(Q_j+Q'_j) + \frac{\xi_j^2}{4\alpha}\right)\right\}$$
$$\exp\left\{\left(iP'_j(Q'_{j+1}-Q'_j) - \frac{\epsilon}{2}({P'}_j^2+\omega^2 {Q'}_j^2)\right)\right\} =$$
$$= \int \frac{dQ_j\, dQ'_j}{2\pi\epsilon} \exp\left\{-\epsilon\left(\frac{\alpha}{4}(Q_j-Q'_j)^2 + \frac{i}{2}\xi_j(Q_j+Q'_j) + \frac{\xi_j^2}{4\alpha}\right)\right.$$
$$\left. + \frac{i}{2}\left(\frac{1}{\epsilon}(Q_{j+1}-Q_j)^2 - \epsilon\omega^2 Q_j^2\right) - \frac{i}{2}\left(\frac{1}{\epsilon}(Q'_{j+1}-Q'_j)^2 - \epsilon\omega^2 {Q'}_j^2\right)\right\} \cdot$$
$$\cdot \langle Q_j, t_j|\hat\rho_{[\xi]}(t_j)|Q'_j, t_j\rangle\,. \tag{65}$$

Iterating eq. (65) we obtain finally for the characteristic operator

$$\langle Q_F, t_F|\mathcal{G}(t_F,\, t_0;\, [\xi])\hat\rho_0|Q'_F, t_F\rangle \equiv \langle Q_F, t_F|\hat\rho_{[\xi]}(t_F)|Q'_F, t_F\rangle =$$
$$= \int dQ_0 \int dQ'_0\; \langle Q_0, t_0|\hat\rho_0|Q' t_0\rangle \int \mathcal{D}Q \int \mathcal{D}Q' \tag{66}$$
$$\exp \sum_{j=0}^{N-1} \left\{-\epsilon\left(\frac{\alpha}{4}(Q_j-Q'_j)^2 + \frac{i}{2}\xi_j(Q_j+Q'_j) + \frac{\xi_j^2}{4\alpha}\right) + \right.$$
$$\left. + \frac{i}{2}\left(\frac{1}{\epsilon}(Q_{j+1}-Q_j)^2 - \epsilon\omega^2 Q_j^2\right) - \frac{i}{2}\left(\frac{1}{\epsilon}(Q'_{j+1}-Q'_j)^2 - \epsilon\omega^2 {Q'}_j^2\right)\right\},$$

where we have set $Q_N = Q_F,\; Q'_N = Q'_F$,

$$\mathcal{D}Q = \left(\frac{1}{2\pi\epsilon}\right)^{\frac{N}{2}} \prod_{j=1}^{N-1} dQ_j \tag{67}$$

(the usual Feynman measure) and similarly for $\mathcal{D}Q'$ .

Having in mind the ideal limit $N \to \infty$, we can also write in the continuous notation

$$\langle Q_F, t_F|\mathcal{G}(t_F,\, t_0;\, [\xi])\hat\rho_0|Q'_F, t_F\rangle = \tag{68}$$
$$= \int dQ_0 \int dQ'_0\; \langle Q_0, t_0|\hat\rho_0|Q'_0, t_0\rangle \int_{Q_0}^{Q_F} \mathcal{D}Q \int_{Q'_0}^{Q'_F} \mathcal{D}Q'$$
$$\exp \int_{t_0}^{t_F} dt \left\{-\left(\frac{\alpha}{4}(Q-Q')^2 + \frac{i}{2}\xi(Q+Q') + \frac{\xi^2}{4\alpha}\right)\right.$$

$$+\frac{i}{2}\left(\dot{Q}^2-\omega^2 Q^2\right)-\frac{i}{2}\left(\dot{Q}'^{\,2}-\omega^2 Q'^{\,2}\right)\Big\}\ ,$$

where by the extremes in the functional integral we mean that the integration is performed keeping the initial and final values of $Q(t)$ and $Q'(t)$ fixed, i. e. under the condition $Q(t_0)=Q_0\,,\ Q(t_F)=Q_F$ and $Q'(t_0)=Q'_0\,,\ Q'(t_F)=Q'_F$. Note that (66) is the path integral equation corresponding to (32).

Let us then calculate the operation $\mathbf{f}(t_F,t_0;[q(t)])$ according eq. (40), $q(t)$ being now the macroscopic position of the oscillator

$$\langle Q_F,t_F|\mathbf{f}(t_F,t_0;[q(t)])\rho_0|Q'_F,t_F\rangle = \tag{69}$$

$$=\int\mathcal{D}_c\xi\exp\left\{i\int_{t_0}^{t_F}dt\xi(t)\,q(t)\right\}\langle Q_F,t_F|\mathcal{G}(t_F,\,t_0;\,[\xi])\hat\rho_0|Q'_F,t_F\rangle =$$

$$=\int dQ_0\int dQ'_0\ \langle Q_0,t_0|\rho_0|Q'_0,t_0\rangle\int_{Q_0}^{Q_F}\mathcal{D}Q\int_{Q'_0}^{Q'_F}\mathcal{D}Q'\left(\frac{\epsilon}{2\pi}\right)^{\frac{N-1}{2}}\int\prod_{l=1}^{N-1}d\xi_l$$

$$\exp\sum_{j=0}^{N-1}\left\{i\epsilon\xi_j q_j-\epsilon\left(\frac{\alpha}{4}(Q_j-Q'_j)^2+\frac{i}{2}\xi_j(Q_j+Q'_j)+\frac{\xi_j^2}{4\alpha}\right)+\right.$$

$$\left.+\frac{i}{2}\left(\frac{1}{\epsilon}(Q_{j+1}-Q_j)^2-\epsilon\omega^2Q_j^2\right)-\frac{i}{2}\left(\frac{1}{\epsilon}(Q'_{j+1}-Q'_j)^2-\epsilon\omega^2Q'^{\,2}_j\right)\right\}=$$

$$=(2\alpha)^{\frac{N}{2}}\int dQ_0\int dQ'_0\ \langle Q_0,t_0|\hat\rho_0|Q'_0,t_0\rangle\int_{Q_0}^{Q_F}\mathcal{D}Q\int_{Q'_0}^{Q'_F}\mathcal{D}Q'$$

$$\exp\sum_{j=0}^{N-1}\left\{-\frac{\alpha\epsilon}{2}\left((q_j-Q_j)^2+(q_j-Q'_j)^2\right)\right.$$

$$\left.+\frac{i}{2}\left(\frac{1}{\epsilon}(Q_{j+1}-Q_j)^2-\epsilon\omega^2Q_j^2\right)-\frac{i}{2}\left(\frac{1}{\epsilon}(Q'_{j+1}-Q'_j)^2-\epsilon\omega^2Q'^{\,2}_j\right)\right\}$$

or, in the continuous limit,

$$\langle Q_F,t_F|\mathbf{f}(t_F,t_0;[q(t)])\rho_0|Q'_F,t_F\rangle =$$

$$=C_\alpha\int dQ_0\int dQ'_0\ \langle Q_0,t_0|\rho_0|Q'_0,t_0\rangle\int_{Q_0}^{Q_F}\mathcal{D}Q\int_{Q'_0}^{Q'_F}\mathcal{D}Q'$$

$$\exp\int_{t_0}^{t_F}dt\left\{-\frac{\alpha}{2}\left((q-Q)^2+(q-Q')^2\right)\right.$$

$$\left.+\frac{i}{2}\left(\dot{Q}^2-\omega^2Q^2\right)-\frac{i}{2}\left(\dot{Q}'^2-\omega^2Q'^2\right)\right\}\ , \tag{70}$$

where $C_\alpha$ is a normalization constant formally infinite in the limit $N \to \infty$ , which may be possibly incorporated in the classic measure ($\mathcal{D}'_c q = (2\alpha)^{\frac{N}{2}} \mathcal{D}_c q = (\frac{\alpha\epsilon}{\pi})^{\frac{N}{2}} dq_0\, dq_1 \dots dq_{N-1}$).

In operator notation we can also write

$$\mathbf{f}(t_F, t_0; [q(t)])\hat{\rho}_0 = C_\alpha \;\mathrm{T}\exp\left[-\frac{\alpha}{2}\int_{t_0}^{t_1} dt\,(q(t) - \hat{Q}(t))^2\right]\hat{\rho} \\ \cdot \mathrm{T}^\dagger \exp\left[-\frac{\alpha}{2}\int_{t_0}^{t_1} dt\,(q(t) - \hat{Q}(t))^2\right]. \tag{71}$$

Eq. (70) or (71) shows that $\mathbf{f}(t_F, t_0; [q(t)])$ is a (completely) positive mapping.

Notice that eqs. (50) and (52) become

$$\langle q(t)\rangle = \mathrm{Tr}\left\{\hat{Q}(t)\mathcal{G}(t, t_0)\hat{\rho}_0\right\} \tag{72}$$

and

$$\langle q(t)q(t')\rangle = \frac{1}{2\alpha}\delta(t - t') \tag{73}$$
$$+\,\theta(t - t')\frac{1}{2}\mathrm{Tr}\left(\hat{Q}(t)\,\mathcal{G}(t, t')\{\hat{Q}(t'),\, \mathcal{G}(t',\, t_0)\hat{\rho}_0\}\right)$$
$$+\,\theta(t' - t)\frac{1}{2}\mathrm{Tr}\left(\hat{Q}(t')\,\mathcal{G}(t', t)\{\hat{Q}(t),\, \mathcal{G}(t,\, t_0)\hat{\rho}_0\}\right).$$

Then, let us introduce the quantity (cf. eq. (53))

$$q_h(t) = \int dt''\, h(t - t'')\, q(t'') \tag{74}$$

and set

$$\langle \hat{A}(t)\rangle_{\mathrm{QM}} = \mathrm{Tr}\left(\hat{A}(t)\mathcal{G}(t, t_0)\hat{\rho}_0\right), \tag{75}$$

as the most direct analogous quantity to the ordinary Quantum Mechanics expectation value. Typical assumptions for $h(t)$ may be

$$h(t) = \frac{1}{\tau}\chi_{(-\frac{\tau}{2},\, \frac{\tau}{2})}(t)\,, \qquad \frac{1}{\tau\sqrt{\pi}}e^{-\frac{t^2}{\tau^2}}\,, \tag{76}$$

$\chi_{(-\frac{\tau}{2},\, \frac{\tau}{2})}(t)$ being the characteristic function of the interval $(-\frac{\tau}{2},\, \frac{\tau}{2})$ .

We have

$$\langle q_h(t) \rangle = \langle \hat{Q}_h(t) \rangle_{\rm QM} \tag{77}$$

and, e. g. for the first choice,

$$\langle (q_h(t) - \langle q_h(t) \rangle)^2 \rangle = \frac{1}{2\alpha\tau} + \left\langle \left( \hat{Q}_h(t) - \langle \hat{Q}_h(t) \rangle_{\rm QM} \right)^2 \right\rangle_{\rm QM} , \tag{78}$$

in case $\tau$ is so small that $\mathcal{G}(t,t')$ can be replaced by the identity for $t - t' < \tau$.

The first term in (73) or (78) has no counterpart in ordinary quantum theory and is what we have called the *intrinsic* part of the variance is the introduction.

### 3.2 Real scalar field

To extend the formalism of sect. 2 to fields and make the theory relativistic covariant, eq. (31) has to be replaced by

$$\frac{\delta}{\delta\sigma(x)} \mathcal{G}(\sigma, \sigma_0; [j(x)]) = \mathcal{K}(x,\, j(x)) \mathcal{G}(\sigma, \sigma_0; [j(x)]) \,, \tag{79}$$

where $x \equiv (t, \mathbf{x})$, $\sigma$ and $\sigma_0$ are *spacelike* hypersurfaces and $\mathcal{K}(x,\, j(x))$ is expressed only in terms of field operators in the point $x$.

Note that, in order the above equation to be consistent, the following condition has to be satisfied on $\sigma$

$$\mathcal{K}(x,\, j(x)) \mathcal{K}(x',\, j(x')) = \mathcal{K}(x',\, j(x')) \mathcal{K}(x,\, j(x)) \,. \tag{80}$$

Furthermore, identifying $\sigma$ and $\sigma_0$ with time constant hyperplanes and integrating over the space coordinates, eq. (79) becomes

$$\frac{\partial}{\partial t} \mathcal{G}(t, t_0; [j]) = \int d^3\mathbf{x}\, \mathcal{K}(t, \mathbf{x};\, j(x)) \mathcal{G}(t, t_0; [j]) \,, \tag{81}$$

which is of the general form (31) with $\mathbf{x}$ playing the role of a component index (cf. eq. (44).

Then let us consider the case of a free real scalar field with density of Lagrangian

$$L(x) = \frac{1}{2} (\partial_\mu \varphi(x)\, \partial^\mu \varphi(x) - m^2\, \varphi^2(x)) \,. \tag{82}$$

The canonical equal time commutation relation are

$$[\hat{\varphi}(x), \hat{\varphi}(x')]_{\mathrm{ET}} = 0, \qquad [\hat{\varphi}(x), \hat{\pi}(x')]_{\mathrm{ET}} = i\delta(\vec{x}-\vec{x}'), \qquad [\hat{\pi}(x), \hat{\pi}(x')]_{\mathrm{ET}} = 0\,, \tag{83}$$

with $\hat{\pi}(x) = \hat{\partial}_0 \varphi(x)$ and the energy-momentum tensor

$$\hat{T}_{\mu\nu}(x) = \partial_\mu \hat{\varphi}(x)\, \partial_\nu \hat{\varphi}(x) - g_{\mu\nu} \hat{L}(x)\,. \tag{84}$$

The *macroscopic* field $\phi(x)$ is introduced by setting

$$\mathcal{L}(x)\, \hat{\rho} = -\frac{\alpha}{4} [\hat{\varphi}(x), [\hat{\varphi}(x), \hat{\rho}]] \tag{85}$$

and

$$\mathcal{K}(x,\, j(x))\, \hat{\rho} = \mathcal{L}(x)\, \hat{\rho} - \frac{i}{2} j(x)\, \{\hat{\varphi}(x), \hat{\rho}\} - \frac{1}{4\alpha} j^2(x) \hat{\rho}\,. \tag{86}$$

It can be checked that the consistency condition is satisfied if

$$[\hat{\varphi}(x),\, \hat{\varphi}(x')] = 0\,, \quad \text{for } x' \text{ out of the light cone of } x\,, \tag{87}$$

what of course follows from eq. (83) and Lorentz invariance.

Eq.(79) or (81) can be integrated in terms of a path integral as in the case of the harmonic oscillator and completely similar developments can be performed. The main difference is that now the space-time region between the initial and final hypersurfaces $\sigma_0$ and $\sigma_F$ has to be divided in four-dimensional cells of side $\epsilon$ (possibly restricting initially the three-dimensional space to a finite volume $V$ and then considering the limit $\epsilon \to 0$ and $V \to \infty$) and correspondingly the time integration has to be replaced by a space-time integration.

So instead of (68) we have

$$\begin{aligned} &\langle \varphi_F, \sigma_F | \mathcal{G}(\sigma_F,\, \sigma_0;\, [j]) \hat{\rho}_0 | \varphi'_F, \sigma_F \rangle = \\ &\qquad \int \mathcal{D}_{\sigma_0} \varphi_0 \int \mathcal{D}_{\sigma_0} \varphi'_{\sigma_0}\ \langle \varphi_0, \sigma_0 | \rho_0 | \varphi'_0, \sigma_0 \rangle \int_{\varphi_0}^{\varphi_F} \mathcal{D}\varphi \int_{\varphi'_0}^{\varphi'_F} \mathcal{D}\varphi' \\ &\exp \int_{\sigma_0}^{\sigma_F} d^4x \left\{ - \left( \frac{\alpha}{4} (\varphi - \varphi')^2 + \frac{i}{2} j (\varphi + \varphi') + \frac{j^2}{4\alpha} \right) \right. \\ &\qquad \left. + \frac{i}{2} \left( \partial_\mu \varphi\, \partial^\mu \varphi - m^2 \varphi^2 \right) - \frac{i}{2} \left( \partial_\mu \varphi'\, \partial^\mu \varphi' - m^2 \varphi'^2 \right) \right\}\,, \end{aligned} \tag{88}$$

where $|\varphi, \sigma\rangle$ are the simultaneous eigenstates of $\hat{\varphi}(x)$ for $x$ on the spacelike hypersurface $\sigma$ ,

$$\hat{\varphi}(x)|\varphi, \sigma\rangle = \varphi(x)|\varphi, \sigma\rangle \qquad \text{for any} \qquad x \in \sigma \,, \tag{89}$$

$\mathcal{D}\varphi$ denotes the functional measure in the space of the $\varphi(x)$s regarded as functions in the four dimensional space and $\mathcal{D}_\sigma\varphi$ the analogous measure for $\varphi(x)$ regarded as functions on the three-dimensional surface $\sigma$, i.e. (for simplicity specifically referring to the case in which $\sigma_0$ and $\sigma_F$ are equal time hyperplanes, $t = t_0$ and $t = t_F$)

$$\mathcal{D}\varphi \simeq \left(\frac{1}{2\pi\epsilon\delta^3}\right)^{N/2} \prod_j d\varphi(x_j) \qquad \mathcal{D}_\sigma\varphi \simeq \left(\frac{1}{2\pi\delta^3}\right)^{N_\sigma/2} \prod_{x_j \in \sigma} d\varphi(x_j) \,, \tag{90}$$

$N$ and $N_\sigma$ being the total number of cells in which the spacetime region delimited by the volume $V$ and the surfaces $\sigma_0$ and $\sigma_F$ is divided and the number of cells intersected by the surface *sigma*, respectively, $\epsilon$ and $\delta$ the time and the space side of each cell.

We have also instead of (71)

$$\begin{aligned} \mathbf{f}(\sigma_F, \sigma_0; [\varphi(x)])\hat{\rho}_0 = C_\alpha \; \mathrm{T} \exp\left[-\frac{\alpha}{2}\int_{\sigma_0}^{\sigma_F} d^4x \, (\phi(x) - \hat{\varphi}(x))^2\right] \cdot \\ \cdot \, \hat{\rho}_0 \, \mathrm{T}^\dagger \exp\left[-\frac{\alpha}{2}\int_{\sigma_0}^{\sigma_F} d^4x \, (\phi(x) - \hat{\varphi}(x))^2\right] . \end{aligned} \tag{91}$$

and

$$\begin{aligned} \langle\phi(x)\rangle = -i\frac{\delta}{\delta j(x)} \mathrm{Tr}\left[\mathcal{G}(\sigma_F, \sigma_0; [j])\hat{\rho}_0\right]|_{j=0} = \\ = \mathrm{Tr}\left[\mathcal{G}(\sigma, \sigma_0)\hat{\rho}_0\right] = \langle\hat{\varphi}(x)\rangle_{\mathrm{QM}} \,, \end{aligned} \tag{92}$$

$\sigma$ being any spacelike surface trough $x$, and furthermore

$$\begin{aligned} \langle\phi(x)\phi(x')\rangle = \frac{1}{2\alpha}\delta^4(x - x') \\ +\theta(t - t')\,\frac{1}{2}\,\mathrm{Tr}\left[\hat{\varphi}(x)\,\mathcal{G}(\sigma, \sigma')\{\varphi(x'),\, \mathcal{G}(\sigma',\, \sigma_0)\hat{\rho}_0\}\right] \\ +\theta(t' - t)\,\frac{1}{2}\,\mathrm{Tr}\left[\hat{\varphi}(x')\,\mathcal{G}(\sigma', \sigma)\{\varphi(x),\, \mathcal{G}(\sigma,\, \sigma_0)\hat{\rho}_0\}\right] . \end{aligned} \tag{93}$$

In place of eq. (74) we have to consider spacetime averages of the type

$$\phi_h(x) = \int d^4x' h(x'-x)\phi(x')\,, \tag{94}$$

where for instance

$$h(x) = \frac{1}{a^4}\chi_\omega(x)\,, \tag{95}$$

$\omega$ being a four-dimensional cube of side $a$ centered on the origin, and for $a$ sufficiently small

$$\left\langle (\phi_h(x) - \langle\phi_h(x)\rangle)^2 \right\rangle = \frac{1}{2\alpha a^4} + \left\langle (\hat\varphi_h(x) - \langle\hat\varphi_h(x)\rangle_{\text{QM}})^2 \right\rangle_{\text{QM}}\,. \tag{96}$$

# 4 Problems with the conservation laws

At this point we may note that equations of the type (61) and (85) are in conflict with the law of energy conservation, what does not seems acceptable in the prospective we are pursuing of a modification of ordinary Quantum Theory.

In the present framework, for energy conservation we intend the conservation of $\langle\hat H\rangle_{\text{QM}}$

In fact for any given observable one has

$$\begin{aligned}\frac{d}{dt}\langle\hat A(t)\rangle_{\text{QM}} &= \text{Tr}\left[\frac{d\hat A}{dt}\,\mathcal{G}(t,t_0)\,\hat\rho\right] + \text{Tr}\left[\hat A(t)\frac{\partial}{\partial t}\,\mathcal{G}(t,t_0)\hat\rho\right] = \\ &= \langle\frac{d\hat A(t)}{dt}\rangle_{\text{QM}} + \text{Tr}\left[\hat A(t)\,\mathcal{L}(t)\,\mathcal{G}(t,t_0)\hat\rho\right] = \\ &= \langle\frac{d\hat A(t)}{dt}\rangle_{\text{QM}} + \text{Tr}\left[(\mathcal{L}'(t)\hat A(t))\,\mathcal{G}(t,t_0)\hat\rho\right]\,,\end{aligned} \tag{97}$$

i. e.

$$\frac{d}{dt}\langle\hat A(t)\rangle_{\text{QM}} = \langle\frac{d\hat A(t)}{dt} + \mathcal{L}'(t)\hat A(t)\rangle_{\text{QM}}\,, \tag{98}$$

where by $\mathcal{L}'$ we denote the dual mapping of $\mathcal{L}$ (see (10)).

For the harmonic oscillator we have

$$\mathcal{L}'(t)\hat A(t) = -\frac{\alpha}{4}[[\hat A(t),\hat Q(t)],\hat Q(t)] = -\frac{\alpha}{4}[\hat Q(t)[\hat Q(t),\hat A(t)]]\,. \tag{99}$$

Consequently, since obviously $\frac{d}{dt}\hat{H} = 0$,

$$\frac{d}{dt}\langle \hat{H} \rangle_{\rm QM} = \langle \mathcal{L}'(t)\hat{H} \rangle_{\rm QM} = \frac{\alpha}{4} \neq 0 \tag{100}$$

and the energy is no longer conserved.

In a similar way for the real scalar field

$$\begin{aligned} &\partial_\mu \langle \hat{T}^{\mu\,\nu}(t,\mathbf{x}) \rangle_{\rm QM} = \\ &= \langle \partial_\mu \hat{T}^{\mu\,\nu}(t,\mathbf{x}) + \int d^3\vec{y}\, \mathcal{L}'(t,\mathbf{y})\, \hat{T}^{0\,\nu}(t,\mathbf{x}) \rangle_{\rm QM} = \frac{1}{4} g^{0\,\nu}\, \delta(\mathbf{0})\,, \end{aligned} \tag{101}$$

which is not only different from 0 but even infinite and non-covariant, showing that $\mathcal{L}(x)$ as given by (85) is hill defined.

*At a formal level* the above difficulty can be overcome modifying the definition of $\mathcal{L}(t)$ and $\mathcal{L}(x)$.

For the harmonic oscillator we can replace (61) with

$$\begin{aligned} \mathcal{L}(t) &= -\frac{\gamma}{4}\left( [\hat{P}(t),[\hat{P}(t),\hat{\rho}]] - \omega^2 [\hat{Q}(t),[\hat{Q}(t),\hat{\rho}]] \right) \\ &= -\frac{\gamma}{4}\left( [\dot{\hat{Q}}(t),[\dot{\hat{Q}}(t),\hat{\rho}]] - \omega^2 [\hat{Q}(t),[\hat{Q}(t),\hat{\rho}]] \right)\,, \end{aligned} \tag{102}$$

$\gamma$ being now an adimensional constant. Then we have immediately

$$\mathcal{L}'(t)\hat{H} = 0 \qquad \Rightarrow \qquad \frac{d}{dt}\,\langle \hat{H} \rangle_{\rm QM} = 0\,. \tag{103}$$

Similarly for the scalar field we can take

$$\mathcal{L}(x)\,\hat{\rho} = -\frac{\gamma}{4}([\partial_\mu \hat{\varphi}(x),[\partial^\mu \hat{\varphi}(x),\hat{\rho}]] - m^2[\hat{\varphi}(x),[\hat{\varphi}(x),\hat{\rho}]])\,; \tag{104}$$

and then again

$$\mathcal{L}'(t,\mathbf{y})\,\hat{T}^{0\,\nu}(t,\mathbf{x}) = 0 \qquad \Rightarrow \qquad \partial_\mu \langle \hat{T}^{\mu\,\nu}(x) \rangle_{\rm QM} = 0\,. \tag{105}$$

Note the similarity between the factors multiplying $-\frac{\gamma}{4}$ in (102) and (104) and the corresponding Lagrangian and density of Lagrangian (57) and (84) respectively.

Note also that with the definitions (102) and (104) the mappings $\mathcal{G}(t,t_0)$ and $\mathcal{G}(\sigma,\sigma_0)$ are no longer positive. E. g. we can write

$$\begin{aligned}\mathcal{G}(t+\epsilon,t)\hat{\rho} &= \hat{\rho}-\frac{\epsilon\gamma}{4}([\hat{P}(t),[\hat{P}(t),\hat{\rho}]]-\omega^2[\hat{Q}(t),[\hat{Q}(t),\hat{\rho}]]) = \\ &= \{1-\frac{\epsilon\gamma}{4}(\hat{P}^2-\omega^2\hat{Q}^2\}\{\hat{\rho}+\frac{\epsilon\gamma}{2}(\hat{P}\hat{\rho}\hat{P}-\omega^2\hat{Q}\hat{\rho}\hat{Q})\}\{1-\frac{\epsilon\gamma}{4}(\hat{P}^2-\omega^2\hat{Q}^2)\}\end{aligned} \tag{106}$$

and in the last line the middle factor is not positive in general.

However we shall show that the definition of $\mathcal{K}(t,[\xi])$ and $\mathcal{K}(x,[j])$ can be modified in a corresponding way that the densities $\mathbf{f}(t_F,t_0;[q])$ and $\mathbf{f}(\sigma_F,\sigma_0;[\phi])$ remain positive.

# 5 Modified formalism

## 5.1 Harmonic oscillator

According to (102) we assume

$$\mathcal{L}(t)\hat{\rho} = -\frac{\gamma}{4}([\hat{P}(t),[\hat{P}(t),\hat{\rho}]]-\omega^2[\hat{Q}(t),[\hat{Q}(t),\rho]]) \tag{107}$$

and correspondingly take

$$\begin{aligned}\mathcal{K}(t,\xi)\hat{\rho} = \mathcal{L}(t)\hat{-}\frac{i}{2}\xi(t)\{\hat{Q}(t),\hat{\rho}\,\} \\ -\frac{1}{2\gamma}\int_{t_0}^{t} dt'\,G(t-t')\,\xi(t)\,\xi(t')\,\hat{\rho}\,,\end{aligned} \tag{108}$$

where

$$G(t-t') = \frac{1}{T}\sum_{n=-\infty}^{\infty}\frac{1}{k_n^2-\omega^2}\,e^{ik_n(t-t')} \simeq \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,\mathrm{P}\frac{1}{k^2-\omega^2}\,e^{ik(t-t')} \tag{109}$$

($T=t_F-t_0$, $\;k_n=\frac{2\pi n}{T}$, $\;n=0,\,\pm1,\,\pm2\,,\ldots$) is the solution of the equation

$$\mathrm{K}\,G(t-t') \equiv -(\frac{d^2}{dt^2}+\omega^2)\,G(t-t') = \delta(t-t') \tag{110}$$

i.e. it is the inverse of the differential operator K . [2] Obviously we can also write

$$
\begin{aligned}
&K(t-t') \equiv \mathrm{K}\delta(t-t') = \\
&\quad \frac{1}{T}\sum_{n=-\infty}^{\infty}(k_n^2-\omega^2)\, e^{ik_n(t-t')} \simeq \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,(k^2-\omega^2)\, e^{ik(t-t')} \,. \quad (111)
\end{aligned}
$$

Note that, with our modified definition, $\mathcal{K}(t,\xi)$ depends not only on the value of $\xi$ at the time $t$, as in (64), but on its the entire history before $t$. Even with $\mathcal{G}(t_b,\, t_a;\, [\xi]))$ defined as in (32), therefore, it has to be equally understood that the integral in $t'$ is extended from $t_0$ to $t$. Then eq. (29) and consequently

$$
\mathcal{G}(t_c,t_b)\,\mathcal{G}(t_b,t_a) = \mathcal{G}(t_c,t_a) \qquad (112)
$$

remain formally valid. On the contrary, strictly speaking Eq. (23) is no longer an exact equation; however, as we shall see, it remains valid as a good approximation for all practical purposes.

According (107, 108) now instead of (64) we have

$$
\begin{aligned}
&\hat\rho_{[\xi]}(t_{i+1}) = \hat\rho_{[\xi]}(t_i) - \qquad (113)\\
&-\epsilon\left\{\frac{\gamma}{4}\left([\hat P(t_i),[\hat P(t_i),\hat\rho_{[\xi]}(t_i)]] - \omega^2[\hat Q(t_i),[\hat Q(t_i),\hat\rho_{[\xi]}(t_i)]]\right) + \right.\\
&\qquad \left. +\frac{i}{2}\xi(t)\{\hat Q(t_i),\, \hat\rho_{[\xi]}(t_i)\} + \frac{1}{2\gamma}\int_{t_0}^{t_i} dt' G(t_i-t')\,\xi(t_i)\xi(t')\,\hat\rho_{[\xi]}(t_i)\right\}
\end{aligned}
$$

and, by appropriately inserting completenesses in the momentum eigenvectors,

$$
\begin{aligned}
&\langle Q_{i+1},\, t_{i+1}|\hat\rho_{[\xi]}(t_{i+1})|Q'_{i+1},\, t_{i+1}\rangle = \\
&\quad = \int\frac{dQ_i\, dP_i}{2\pi}\int\frac{dQ'_i\, dP'_i}{2\pi}\exp\left\{i\left[P_i(Q_{i+1}-Q_i) - \frac{\epsilon}{2}(P_i^2-\omega^2 Q_i^2)\right]\right\}\\
&\qquad\qquad \exp\left\{-\frac{\epsilon\gamma}{4}\left[(P_i-P'_i)^2 - \omega^2(Q_i-Q'_i)^2\right] - \right.
\end{aligned}
$$

[2] In the expression as an integral in eq. (109) some kind of regularization has to be assumed. At the discrete level this amounts to make a choice for the ratio between $T$ and the classic period $\frac{2\pi}{\omega}$, in order to avoid that some $k_n$ coincides with $\pm\omega$. The principal value prescription obviously corresponds to assume $\frac{\omega T}{2\pi}$ to be an half odd integer. However, since actually,as we shall see, we can restrict ourselves to values of $|k_n| > \omega$ the specific choice turns out to be irrelevant

$$
\begin{aligned}
&\qquad -\frac{i\epsilon}{2}\xi_i(Q_i+Q_i') - \frac{\epsilon^2}{2\gamma}\sum_{j=0}^{i}{}' \, G_{i\,j}\,\xi_i\xi_j\Bigg\} \\
&\exp\left\{ i\left[P_i'(Q_{i+1}'-Q_i') - \frac{\epsilon}{2}(P_i'^{\,2}-\omega^2 Q_i'^{\,2})\right]\right\}\langle Q_i,\, t_i|\rho_{[\xi]}|Q_i',\, t_j\rangle = \\
&= \frac{1}{2\pi\epsilon}\int dQ_i \int dQ_i' \exp\left\{-\frac{\gamma}{4}\left[\frac{1}{\epsilon}\left((Q_{i+1}-Q_i)-\right.\right.\right. \\
&\qquad \left.\left. -(Q_{i+1}'-Q_i')\right)^2 - \epsilon\omega^2(Q_i-Q_i')^2\right] + \\
&+\frac{i}{2}\left(\frac{1}{\epsilon}(Q_{i+1}-Q_i)^2 - \epsilon\,\omega^2 Q_i^2\right) - \frac{i}{2}\left(\frac{1}{\epsilon}(Q_{i+1}'-Q_i')^2 - \epsilon\omega^2 Q_i'^{\,2}\right) - \\
&\qquad \left. -\frac{i\epsilon}{2}\xi_i(Q_i+Q_i') - \frac{\epsilon^2}{2\gamma}\sum_{j=0}^{i}{}' \, G_{ij}\,\xi_i\xi_j\right\}\langle Q_i,\, t_i|\hat\rho_{G[\xi]}(t_i)|Q_i',\, t_i\rangle\,,
\end{aligned}
\tag{114}
$$

where $G_{ij}$ stays for an appropriate discretization of $G(t-t')$, to be specified later, and the prime in the sum indicates that the diagonal elements $G_{ii}$ must to be multiplied for $\frac{1}{2}$ so that by exploiting the symmetry of such quantity we can write $\sum_{i=0}^{N-1}\sum_{j=0}^{i\,\prime} G_{i\,j}\,\xi_i\,\xi_j = \frac{1}{2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} G_{i\,j}\xi_i\,\xi_j$.

Iterating eq. (114) over the entire interval $(t_0,\, t_F)$ with the same definitions as in (67), we obtain

$$
\begin{aligned}
&\langle Q_F, t_F|\mathcal{G}(t_F,\, t_0;\, [\xi])\hat\rho_0|Q_F', t_F\rangle = \\
&= \int dQ_0 \int dQ_0' \,\langle Q_0, t_0|\hat\rho_0|Q' t_0\rangle \int_{Q_0}^{Q_F}\mathcal{D}Q \int_{Q_0'}^{Q_F'}\mathcal{D}Q' \\
&\exp\sum_{i=0}^{N-1}\left\{-\frac{\gamma}{4}\left[\frac{1}{\epsilon}\left((Q_{i+1}-Q_i)-(Q_{i+1}'-Q_i')\right)^2 - \epsilon\omega^2(Q_i-Q_i')^2\right]\right. \\
&\qquad +\frac{i}{2}\left(\frac{1}{\epsilon}(Q_{i+1}-Q_i)^2 - \epsilon\omega^2 Q_i^2\right) - \frac{i}{2}\left(\frac{1}{\epsilon}(Q_{i+1}'-Q_i')^2 - \epsilon\omega^2 Q_i'^{\,2}\right) \\
&\qquad\qquad \left. -\frac{i\epsilon}{2}\xi_i(Q_i+Q_i') - \frac{\epsilon^2}{2\gamma}\sum_{j=0}^{i}{}' \, G_{i\,j}\xi_i\xi_j\right\} = \\
&= \int dQ_0 \int dQ_0' \,\langle Q_0, t_0|\hat\rho_0|Q' t_0\rangle \int_{Q_0}^{Q_F}\mathcal{D}Q \int_{Q_0'}^{Q_F'}\mathcal{D}Q' \\
&\exp\left\{\sum_{i,j=0}^{N}\frac{1}{\epsilon}K_{i\,j}\left[-\frac{\gamma}{4}(Q_i-Q_i')(Q_j-Q_j') + \frac{i}{2}Q_iQ_j - \frac{i}{2}Q_i'Q_j'\right]\right.
\end{aligned}
$$

$$-\frac{i\epsilon}{2}\sum_{j=0}^{N-1}\xi_j(Q_j+Q'_j)-\frac{\epsilon^2}{4\gamma}\sum_{i,j=0}^{N-1}\xi_i G_{i\,j}\xi_j\Bigg\}\ , \tag{115}$$

where we have set again $Q_N = Q_F$ and $Q'_N = Q'_F$ and denoted by $K_{i\,j}$ the $(N+1)\times(N+1)$ matrix

$$K_{i\,j}=\begin{pmatrix} 1-\epsilon^2\omega^2 & -1 & 0 & \dots & 0 & 0\\ -1 & 2-\epsilon^2\omega^2 & -1 & \dots & 0 & 0\\ 0 & -1 & 2-\epsilon^2\omega^2 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 2-\epsilon^2\omega^2 & -1\\ 0 & 0 & 0 & \dots & -1 & 1 \end{pmatrix}\ , \tag{116}$$

which obviously provides a discretization of (111).

Now, let us denote by $\bar{K}_{ij}$ the $N\times N$ matrix obtained by suppressing the last row and column in $K_{ij}$. For $N$ sufficiently large ($\epsilon$ small) $\bar{K}_{ij}$ turns out to be positive (App. B) Furthermore, while $\det K_{ij}=0|_{\epsilon=0}$, it is $\det \bar{K}_{ij}|_{\epsilon=0}=1$. Then we can chose $G_{ij}=\epsilon\bar{K}^{-1}_{ij}$, which is also a positive matrix, and we have

$$\left(\frac{\epsilon}{2\pi}\right)^{\frac{N}{2}}\int d\xi_0\dots d\xi_{N-1}\,\exp\left\{i\epsilon\sum_{j=0}^{N-1}\xi_j\,\left(q_j-\frac{Q_j+Q'_j}{2}\right)-\frac{\epsilon}{4\gamma}\sum_{i,j=0}^{N-1}\xi_i G_{ij}\xi_j\right\}$$
$$=\left(\frac{2\gamma}{\epsilon^2}\right)^{\frac{N}{2}}\exp\left\{-\frac{\gamma}{\epsilon}\sum_{i,j=0}^{N-2}\left(q_i-\frac{Q_i+Q'_i}{2}\right)\bar{K}_{ij}\left(q_j-\frac{Q_j+Q'_j}{2}\right)\right\}\ . \tag{117}$$

Finally, by similar manipulations to those performed in (69), we obtain

$$\langle Q_F,t_F|\mathbf{f}(t_F,\,t_0;\,[q(t)])\,\hat{\rho}_0|Q'_F,t_F\rangle=\left(\frac{2\gamma}{\epsilon^2}\right)^{\frac{N}{2}}\int dQ_0\,dQ'_0$$
$$\int_{Q_0}^{Q_F}\mathcal{D}Q\,\exp\sum_{j=0}^{N-2}\left\{-\frac{\gamma}{2}\left[\frac{1}{\epsilon}\left((q_{i+1}-Q_{i+1})-(q_i-Q_i)\right)^2\right.\right.$$
$$\left.\left.-\,\omega^2\epsilon(q_i-Q_i)^2\right]+\frac{i}{2}\left[\frac{1}{\epsilon}(Q_{i+1}-Q_i)^2-\omega^2Q_i^2\right]\right\}\langle Q_0|\hat{\rho}_0|Q'_0\rangle$$
$$\int_{Q'_0}^{Q'_F}\mathcal{D}Q'\,\exp\sum_{i=0}^{N-2}\left\{-\frac{\gamma}{2}\left[\frac{1}{\epsilon}\left((q_{i+1}-Q'_{i+1})-(q_i-Q'_i)\right)^2-\right.\right.$$

$$
-\omega^2\epsilon(q_i - Q'_i)^2\Big] + \frac{i}{2}\left[\frac{1}{\epsilon}(Q'_{i+1} - Q'_j)^2 - \omega^2\epsilon Q'^2_j\right]\Bigg\} \qquad (118)
$$

$$
\exp\left\{-\frac{\gamma}{4\epsilon}\,\left[(Q_F - Q'_F)^2 - 2(Q_F - Q'_F)(Q_{N-1} - Q'_{N-1})\right]\right\}\,.
$$

Eq. (118) is analogous to eq. (69) for the original formalism apart from the occurrence under the integral of the last exponential factor

$$
\exp\left\{-\frac{\gamma}{4\epsilon}\,\left[(Q_F - Q'_F)^2 - 2(Q_F - Q'_F)(Q_{N-1} - Q'_{N-1})\right]\right\}\,. \qquad (119)
$$

Equivalently, by setting

$$
q_N \equiv q_F = \frac{Q_F + Q'_F}{2}\,, \qquad (120)
$$

we can omit the factor (119) and extend the sum in (118) to $N-1$.

The fact that (119) or $q_F$ in (120) depend on both the primed and unprimed variables prevents us from concluding that $\mathbf{f}(t_F,\, t_0;\, [q(t)])$ is positive. On the other side, when we define the mapping for restrict intervals $\mathbf{f}(t_b,\, t_a;\, [q(t)])$, the factor (119) is essential for the validity of the equation

$$
\mathbf{f}(t_c,\, t_a;\, [q(t)]) \;=\; \mathbf{f}(t_c,\, t_b;\, [q(t)])\,\mathbf{f}(t_b,\, t_a;\, [q(t)])\,, \qquad (121)
$$

corresponding to eq (23). However, for $Q_F = Q'_F$ the factor (119) reduces to 1 and $\mathbf{f}(t_F,\, t_0;\, [q(t)])$ coincides with $\bar{\mathbf{f}}(t_F,\, t_0;\, [q(t)])$, defined by the same eq. (118) as it stays but omitting the factor (119) and, obviously, $\bar{\mathbf{f}}(t_F,\, t_0;\, [q(t)])$ is positive. Then even the functional probability density

$$
p\,(t_F,\, t_0;\, [q(t)]) = \mathrm{Tr}\left\{\mathbf{f}(t_F,\, t_0;\, [q(t)])\,\hat{\rho}_0\right\} = \mathrm{Tr}\left\{\bar{\mathbf{f}}(t_F,\, t_0;\, [q(t)])\hat{\rho}_0\right\} \qquad (122)
$$

is positive and this is what matters.

As in sec. 3, in the limit $N \to \infty$ eqs. (114) and (118) can be formally written

$$
\begin{aligned}
&\langle Q_F, t_F|\mathcal{G}(t_F,\, t_0;\, [\xi])\hat{\rho}_0|Q'_F, t_F\rangle = \\
&= \int dQ_0 \int dQ'_0\;\langle Q_0, t_0|\hat{\rho}_0|Q', t_0\rangle \int_{Q_0}^{Q_F} \mathcal{D}Q \int_{Q'_0}^{Q'_F} \mathcal{D}Q' \\
&\exp \int_{t_0}^{t_F} dt\left\{-\frac{\gamma}{4}\left[(\dot{Q} - \dot{Q}')^2 - \omega^2(Q - Q')^2\right] + \right.
\end{aligned}
$$

$$+\frac{i}{2}(\dot{Q}^2-\omega^2Q^2)-\frac{i}{2}(\dot{Q}'^2-\omega^2Q'^2)-$$

$$-\frac{i}{2}\xi(Q+Q')-\frac{1}{4\gamma}\int_{t_0}^{t_F}dt'\xi(t)\,G(t-t')\,\xi(t')\Bigg\} \qquad (123)$$

and

$$\langle Q_F,t_F|\bar{\mathbf{f}}(t_F,\,t_0;\,[q(t)])\,\hat{\rho}_0|Q'_F,t_F\rangle=\left(\frac{2\gamma}{\epsilon^2}\right)^{\frac{N}{2}}\int dQ_0dQ'_0$$

$$\int_{Q_0}^{Q_F}\mathcal{D}Q\,\exp\int_{t_0}^{t_F}dt'\left\{-\frac{\gamma}{2}\left[(\dot{q}-\dot{Q})^2-\omega^2(q-Q)^2\right]+\right.$$

$$\left.+\frac{i}{2}(\dot{Q}^2-\omega^2Q^2)\right\}\langle Q_0|\hat{\rho}_0|Q'_0\rangle$$

$$\int_{Q'_0}^{Q'_F}\mathcal{D}Q'\exp\int_{t_0}^{t_F}dt'\left\{-\frac{\gamma}{2}\left[(\dot{q}-\dot{Q}')^2-\omega^2(q_j-Q'_j)^2\right]+\right.$$

$$\left.+\,\frac{i}{2}(\dot{Q}'^2-\omega^2Q^2)\right\}\,, \qquad (124)$$

or in operator form

$$\bar{\mathbf{f}}(t_F,\,t_0;\,[q(t)])\,\hat{\rho}_0= \qquad (125)$$

$$=C_\gamma\;\mathrm{T}\exp\left\{-\frac{\gamma}{2}\int_{t_0}^{t_F}dt\,[(\dot{q}(t)-\dot{\hat{Q}}(t))^2-\,\omega^2(q(t)-\hat{Q}(t))^2]\right\}$$

$$\hat{\rho}_0\;\mathrm{T}^\dagger\exp\left\{-\frac{\gamma}{2}\int_{t_0}^{t_F}dt\,[(\dot{q}(t)-\dot{\hat{Q}}(t))^2-\omega^2(q(t)-\hat{Q}(t))^2]\right\}\,.$$

Actually eqs. (123-126) requires some comments.

What grants convergence of the integral in eq. (115), when performed step by step, it is the prevalence of the terms in $1/\epsilon$ in the exponent over the terms in $\epsilon$. This circumstance may be clearly illustrated on the similar case of the usual Feynman expression of the ordinary amplitude $\langle Q_F,\,t_F|Q_0,\,t_0\rangle$ for the harmonic oscillator, which can be calculated exactly and remains valid even for an imaginary mass $m=i\mu$ with positive $\mu$ (App. A). The positivity of the matrices $\bar{K}_{ij}$ and $G_{ij}=\epsilon\bar{K}^{-1}_{ij}$ grants that the convergence of the integrals in (117) and (118) is even absolute. We have

$$\sum_{i,j=1}^{N-1}\xi_i\,G_{ij}\,\xi_j>0\,, \qquad (126)$$

for any $(\xi_1,\, \xi_2,\, \dots\, \xi_{N-1})$, that in the limit would correspond to

$$\int_{t_0}^{t_F} dt \int_{t_0}^{t_F} dt' \xi(t) G(t-t') \xi(t') =$$
$$= \sum_{k=-\infty}^{\infty} \tilde{\xi}_k^* \frac{1}{k^2-\omega^2}\, \tilde{\xi}_k \sim \int_{\infty}^{\infty} dk\, \tilde{\xi}^*(k)\, \mathrm{P}\frac{1}{k^2-\omega^2}\, \tilde{\xi}(k) > 0\,, \quad (127)$$

where $\tilde{\xi}_k$ 's are the Fourier coefficients and $\tilde{\xi}(k) \sim \sqrt{\frac{2\pi}{T}}\, \tilde{\xi}_k$. the Fourier transform.

Such circumstances are somewhat surprising. The kernel $K(t-t') = -(\frac{d^2}{dt^2} - \omega^2)\delta(t-t')$ and its inverse $G(t-t')$ are not positive in $L^2(t_0,\, t_F)$ and the meaning of eq, (123-126) seems to be questionable. Actually this shows that the specific limit procedure by which the functional integrals have been defined and the continuous kernel $G(t-t')$ approached is crucial. Specifically it implies that only the Fourier components of $Q(t) - Q_c(t)$, $Q'(t) - Q'_c(t)$ and $\xi(t)$ with $|k| > \omega$ give non vanishing contribution in the integrals, $Q_c(t)$ and $Q'_c(t)$ denoting the solutions of the classical equation of motion satisfying the conditions

$$Q_c(t_0) = Q_0\,, \qquad\qquad Q_c(t_F) = Q_F \quad (128)$$

and

$$Q'_c(t_0) = Q'_0\,, \qquad\qquad Q'_c(t_F) = Q'_F\,. \quad (129)$$

This fact can be also explicit checked by considering the continuous function $Q(t)$ and $Q(t')$ obtained by interpolating linearly the discrete values $Q_0,\, Q_1,\, \dots\, Q_F$ and $Q'_0,\, Q'_1,\, \dots\, Q'_F$ respectively (App. C); their components with $|k| < \omega$ vanish for $\epsilon \to 0$.

Alternatively, to be able to proceed directly in the more appealing continuous formalism, by using only general properties of the functional integral, it is convenient to assume explicitly that all the functions of interest, $Q(t)$, $Q'(t)$, $\xi(t)$ (and consequently $q(t)$) are restricted to the subspace with Fourier components with $|k| \geq \omega$ and that the boundary conditions.

$$\xi(t_0) = \xi(t_F) = 0 \quad (130)$$

and

$$q(t_0) = \frac{1}{2}(Q_0 + Q'_0), \qquad\qquad q(t_F) = \frac{1}{2}(Q_F + Q'_F)\,. \quad (131)$$

hold.[3] In fact a crucial point is the integral (117) in which the translational invariance of the measure $\mathcal{D}_c\xi(t)$ has been used.

In practice, once that eq. (123) has been established, the milder assumption expressed by eq. (127) is sufficient for what concerns $\xi(t)$. This enables us to avoid complicate discussions about the behaviour of $\tilde{\xi}(k)$ for $k \sim \omega$, to exploit the principal value prescription in (109) and to include the border value $k = \omega$ and so the classical solutions explicitly in considerations.

Now notice that, according to Eq. (50), we have $\langle q(t)\rangle = \langle \hat{Q}(t)\rangle_{\mathrm{QM}}$ and since $\mathcal{L}'(t)\,\hat{Q}(t)$ and $\mathcal{L}'(t)\,\hat{P}(t)$ vanish, as it can be immediately checked, we have, as in ordinary Quantum Mechanics,

$$\frac{d}{dt}\langle \hat{Q}(t)\rangle_{\mathrm{QM}} = \langle \hat{P}(t)\rangle_{\mathrm{QM}}\,, \qquad\qquad \frac{d}{dt}\langle \hat{P}(t)\rangle_{\mathrm{QM}} = -\omega^2\langle \hat{Q}(t)\rangle_{\mathrm{QM}} \tag{132}$$

So $\langle q(t)\rangle$ is an exact solution of the classical equation of motion as expected, i.e.

$$\langle q(t)\rangle = \langle \hat{Q}(t)\rangle_{QM} = C\,\cos(\omega t + \delta)\,. \tag{133}$$

Furthermore, if we introduce the time average (cf. eq. (74))

$$q_h(t) = \int dt' h(t-t') q(t') \tag{134}$$

we can write

$$\langle q_h(t)\rangle = -i\frac{\partial}{\partial k}\mathrm{Tr}\left\{\mathcal{G}(t_F, t_0; [kh])\hat{\rho}_0\right\}|_{k=0} = \langle Q_h(t)\rangle_{\mathrm{QM}} \tag{135}$$

and, under the assumption of an effective support of $h(t)$ sufficiently small as in eq. (78),

$$\begin{aligned}\left\langle (q_h - \langle q_h\rangle)^2\right\rangle &= -\frac{\partial^2}{\partial k^2}\mathrm{Tr}\left\{\mathcal{G}(t_F, t_0; [kh])\rho_0\right\}|_{k=0} - \langle q_h\rangle^2 = \\ &= \frac{G_{hh}}{\gamma} + \langle (q_h - \langle Q_h\rangle)^2\rangle_{\mathrm{QM}}\,,\end{aligned} \tag{136}$$

where

$$G_{hh} = \int dt \int dt' h(t)\, G(t-t')\, h(t') = \int dk \tilde{h}^*(k)\, \mathrm{P}\frac{1}{k^2-\omega^2}\,\tilde{h}(k)\,, \tag{137}$$

[3] Note that this is also the condition under which the classic action is actually minimal for $Q(t) = Q_c(t)$

$\tilde{h}(k)$ being the Fourier transform of $h(t)$.

More explicitly, according to eq. (54) we can also write

$$
\begin{aligned}
&p(\bar{q}_1, h_1; \bar{q}_2, h_2; \ldots \bar{q}_l, h_l) = \\
&= \frac{1}{2\pi} \int dk_1 dk_2 \ldots dk_l \, e^{i(k_1 \bar{q}_1 + \ldots k_l \bar{q}_l)} \, \mathrm{Tr} \left\{ \mathcal{G}(t_F, t_0; [k_1 h_1 + \ldots k_l h_l]) \hat{\rho}_0 \right\} = \\
&= \int dQ_F \int dQ_0 \, .dQ'_0 \, \langle Q_0, \, t_0 | \hat{\rho}_0 | Q'_0, \, t_0 \rangle \int_{Q_0}^{Q_F} \mathcal{D}Q \int_{Q'_0}^{Q_F} \mathcal{D}Q' \, \exp \int_{t_0}^{t_F} dt \\
&\left\{ \frac{-\gamma}{4} \left[ (\dot{Q} - \dot{Q}')^2 - \omega^2 (Q - Q')^2 \right] + \frac{i}{2} (\dot{Q}^2 - \omega^2 Q^2) - \frac{i}{2} (\dot{Q}'^2 - \omega^2 Q'^2) \right\} \\
&\left( \frac{\gamma}{\pi} \right)^{\frac{1}{2}} \frac{1}{(\det G_{rs}))^{1/2}} \exp \left\{ -\gamma \sum_{rs} \left( \bar{q}_r - \frac{Q_{h_r} + Q'_{h_r}}{2} \right) \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. G^{-1}_{rs} \left( \bar{q}_s - \frac{Q_{h_s} + Q'_{h_s}}{2} \right) \right\} \,, \qquad (138)
\end{aligned}
$$

with

$$
G_{rs} = \int dt \int dt' \, h_r(t) \, G(t - t') \, h_s(t'). \qquad (139)
$$

Consistently with assumption (127) $h(t)$ and $h_1(t), \ldots h_l(t)$ must be such $G_{hh}$ or the matrix $G_{rs}$ be positive. Taking into account that

$$
\int_{-\infty}^{\infty} dk \, \mathrm{P} \frac{1}{k^2 - \omega^2} = 0 \,, \qquad (140)
$$

it is clear that eq. (76)) does not longer provides correct choices. On the contrary an admissible choice would be

$$
h(t) = A \, e^{-\frac{t^2}{\tau^2}} \cos \bar{k} t \,, \qquad (141)
$$

with $\tau \ll \frac{1}{\omega}$, $\bar{k} > \omega$ and somewhat larger than $\frac{\sqrt{2}}{\tau}$. This is not a positive definite function but it is normalized to 1, if $A = \frac{1}{\tau\sqrt{\pi}} \, e^{\frac{\bar{k}^2 \tau^2}{4}}$ .

Indeed, the Fourier transform of (141) is

$$
\tilde{h}(k) = \frac{1}{\sqrt{8\pi}} \, e^{\frac{\bar{k}^2 \tau^2}{4}} \left( e^{-\frac{\tau^2}{4}(k - \bar{k})^2} + e^{-\frac{\tau^2}{4}(k + \bar{k})^2} \right) = \frac{1}{\sqrt{2\pi}} \, e^{-\frac{\tau^2 k^2}{4}} \cosh \frac{k \bar{k} \tau^2}{2} \,. \qquad (142)
$$

For $\bar{k} > \frac{\sqrt{2}}{\tau}$ this quantity develops two symmetrical maxima which become soon very close to $-\bar{k}$ and $\bar{k}$ (for $\bar{k} = \frac{2}{\tau}$ we have already $k_{\max} \simeq 0.95\bar{k}$) and $G_{hh}$ becomes positive. So the factor $\cos \bar{k}t$ has simply the role of projecting $q(t)$ or equivalently $G(t-t')$ on a manifold in which the Fourier component with $|k| > \omega$ are dominant. Note that, if we introduce explicitly the corresponding restrictions on the spectrum of such functions, for $\bar{k}$ not too large (141) is conceptually equivalent to a pure Gaussian.

For the choice (141) with $\omega\tau \ll 1$ and $\bar{k}\tau$ of the order of few units we find

$$\langle q_h(t) \rangle = \int dt' h(t-t') \langle \hat{Q}(t') \rangle_{QM} =$$
$$= C\, e^{-\frac{\omega^2\tau 2}{4}} \cosh \frac{\bar{k}\omega\tau^2}{2} \cos(\omega t + \alpha) \simeq C \, \cos(\omega t + \alpha)\,. \quad (143)$$

For what concerns $G_{hh}$ and so the intrinsic component of the fluctuation in (136), a rough estimate gives

$$G_{hh} \sim \frac{1}{\bar{k}^2 - \omega^2} \int dk\, \tilde{h}^*(k)\, \tilde{h}(k) = \frac{1}{\tau\sqrt{8\pi}} \frac{e^{\frac{\bar{k}^2\tau^2}{2}} + 1}{\bar{k}^2 - \omega^2}\,. \quad (144)$$

This shows that for large frequency the spectrum of the fluctuations of the macroscopic position $q(t)$ around the solution of the classical equation of motion diverges as the frequency increases; the average $q_h(t)$ over a time interval $\tau$ has the effect of damping such fluctuations. For $\bar{k} \sim 2/\tau$ we have simply $G_{hh} \sim 1.7/\tau\bar{k}^2$.

Finally let us observe that, if in (138, 139) $h_r(t)$ is identified with $h(t-t_r)$ with $h(t)$ given by (141) and $t_r - t_s$ large with respect to $\tau$ , the matrix results positive and nearly diagonal and eq. (23) remain a good approximation.

## 5.2 Scalar Field

We assume again (eq.(104))

$$\mathcal{L}(x)\hat{\rho} = -\frac{\gamma}{4}\left( [\partial_\mu \hat{\varphi}(x), [\partial^\mu \hat{\varphi}(x), \hat{\rho}]] - m^2 [\hat{\varphi}(x), [\hat{\varphi}(x), \hat{\rho}]] \right) \quad (145)$$

and set

$$\mathcal{K}(x,j)\hat{\rho} = \mathcal{L}(x)\hat{\rho} - \frac{i}{2} j(x)\{\hat{\varphi}(x),\hat{\rho}\} - \frac{1}{2\gamma}\int_{\sigma_0}^{\sigma} d^4x'\, G_m(x-x')\, j(x)\, j(x')\,, \int \mathcal{D}Q \int \mathcal{D}Q' \tag{146}$$

where

$$G_m(x-x') = \frac{1}{(2\pi)^4}\int d^4k\, \mathrm{P}\frac{1}{k^2-m^2}\, e^{-ik_\mu (x-x')^\mu} \tag{147}$$

is the solution of the equation

$$(\Box - m^2)\, G_m(x-x') = \delta^4(x-x') \tag{148}$$

and therefore the inverse of the differential operator $(\Box - m^2)$ under the appropriate restrictions.

Then, it can be checked that the compatibility condition (80) is still satisfied and all developments of the above subsection can be repeated with obvious modification, using the functional integrations as defined by eq. (90). In this way, we arrive to the corresponding expression for the CFO, that directly in the continuous notation we can write

$$\begin{aligned}
&\langle \varphi_F,\, \sigma_F|\mathcal{G}(\sigma_F,\sigma_0;[j])\,\hat{\rho}_0|\varphi'_F,\sigma_F\rangle = \\
&= \int \mathcal{D}_{\sigma_0}\varphi_0 \int \mathcal{D}_{\sigma_0}\varphi'\, \langle\varphi_0,\, \sigma_0|\hat{\rho}_0|\varphi'_0,\, \sigma_0\rangle \int_{\varphi_0}^{\varphi_F} \mathcal{D}\varphi \int_{\varphi'_0}^{\varphi'_F} \mathcal{D}\varphi' \\
&\exp \int_{\sigma_0}^{\sigma_F} d^4x \left\{ -\frac{\gamma}{4}\left[\partial_\mu(\varphi-\varphi')\,\partial^\mu(\varphi-\varphi') - m^2(\varphi-\varphi')^2\right] \right. \\
&-\frac{i}{2}\, j(\varphi+\varphi') - \frac{1}{4\gamma}\int_{\sigma_0}^{\sigma_F} d^4x' j(x)\, G_m(x-x')\, j(x') + \\
&\left. +\frac{i}{2}(\partial_\mu\varphi\,\partial^\mu\varphi - m^2\varphi'^2) - \frac{i}{2}(\partial_\mu\varphi'\,\partial^\mu\varphi' - m^2\varphi^2) \right\}\,,
\end{aligned} \tag{149}$$

where the functional integrals are defined as in (90), but for convergence $\epsilon$ has to go to 0 faster than $\delta$. As a consequence, now, only the Fourier components of $\varphi(x)$, $\varphi'(x)$ and $j(x)$ with $k^2 \equiv (k_0^2 - \mathbf{k}^2) > m^2$ give contribution in the limit.In analogy with the preceding case, in practice,we may simply restrict the external source $j(x)$ by the condition

$$\int d^4x\, d^4x'\, j(x)\, G_m(x-x')\, j(x') > 0\,. \tag{150}$$

From eq. (149) we can derive the expression for $\mathbf{f}(\sigma_F, \sigma_0; [\phi(x)])$ as a functional of the classic field $\phi(x)$ . In the operator form we can write

$$\mathbf{f}(\sigma_F, \sigma_0; [\phi(x)])\hat{\rho} = C_\gamma \exp\left\{-\frac{\gamma}{2}\int d^4x \right. \tag{151}$$

$$\left.[\partial_\mu(\phi\,(x) - \hat{\varphi}(x))\,\partial^\mu(\phi(x) - \hat{\varphi}(x)) - m^2(\phi(x) - \hat{\varphi}(x))^2]\right\}\ \hat{\rho}\,\cdot$$

$$\exp\left\{-\frac{\gamma}{2}\int d^4x\,[\partial_\mu(\phi\,(x) - \hat{\varphi}(x))\,\partial^\mu(\phi(x) - \hat{\varphi}(x)) - m^2(\phi(x) - \hat{\varphi}(x))^2]\right\}.$$

Eqs. (92) and (96) have to be replaced with

$$\langle\phi_h(x)\rangle = \langle\hat{\varphi}_h(x)\rangle_{\mathrm{QM}} \tag{152}$$

and

$$\left\langle(\phi_h(x) - \langle\phi_h(x)\rangle)^2\right\rangle = \frac{1}{2\gamma}\,G_{hh}\, +\, \left\langle(\hat{\varphi}_h(x) - \langle\hat{\varphi}_h(x)\rangle)^2\right\rangle_{\mathrm{QM}}\,, \tag{153}$$

with

$$G_{hh} = \int d^4x\,d^4x'\,h(x)\,G_m(x - x')\,h(x')\,, \tag{154}$$

where it must be again $G_{hh} > 0$ and a permitted choice would be e.g.

$$h(x) = A\,e^{-\frac{t^2}{\tau^2} - \frac{\mathbf{x}^2}{a^2}}\,\cos\bar{k}t\,, \tag{155}$$

now with $\bar{k}^2$ sufficiently larger than $m^2$ , $\bar{k} > \frac{\sqrt{2}}{\tau}$ and $A = \frac{1}{\pi^2\tau a^3}\,e^{\frac{1}{4}\bar{k}^2\,\tau^2}$ (note that in natural unity we may significantly assume $\tau \gg a$). With such choice $G_{hh} \sim \frac{1}{4\pi^2\tau a^3}\frac{1}{\bar{k}^2 - m^2}\,(e^{\frac{1}{2}\bar{k}^2\,\tau^2} + 1)$ and, if $\bar{k} \sim \frac{2}{\tau}$, we have $G_{hh} \sim 0.2\,\frac{1}{\tau a^3 \bar{k}^2}$.

# 6 Electromagnetic field

If we want to introduce in a similar way a classical field in the case of the electromagnetic (e.m.) field, the form of $\mathcal{L}(x)$ is uniquely determined by Lorentz and gauge invariance, independently of a requirement of energy-momentum conservation.

We must set

$$\begin{aligned}\mathcal{L}(x)\hat{\rho} \ &= \frac{\gamma}{8}\,[\hat{F}_{\mu\nu}(x),\,[\hat{F}^{\mu\nu}(x), \hat{\rho}\,]] = \\ &= -\frac{\gamma}{4}\,\Big([\hat{E}^i(x),\,[\hat{E}^i(x), \hat{\rho}\,]] - [\hat{B}^i(x),\,[\hat{B}^i(x), \hat{\rho}\,]]\Big)\ ,\end{aligned} \tag{156}$$

where

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$
$$E^i = F_{0i} \qquad\qquad B^i = \frac{1}{2}\epsilon_{ijl}\, F_{jl}\,. \tag{157}$$

Eq. (156) corresponds again to coefficients $\alpha_j$ of both signs in (3) or (38). So we are in a situation similar to the cases considered in sec. 5.

Actually we can set

$$\mathcal{K}\,(x,\, j_\rho(x))\,\hat\rho = \mathcal{L}(x)\hat\rho + \tag{158}$$
$$+\frac{i}{2}\, j_\mu(x)\,\{\hat A^\mu(x),\, \hat\rho\} + \frac{1}{2\gamma}\int_{\sigma_0}^{\sigma} d^4x'\, j_\mu(x)\, G^{\mu\nu}(x-x')\, j_\nu(x')\,\hat\rho\,,$$

where $G^{\mu\nu}(x-x')$ is the Green function relative to the differential operator acting on the potential $A^\mu(x)$ in the equation of motion. This obviously depends on the gauge we use.

If we define $G_0(x-x')$ as in (147) with $m=0$, in the Coulomb Gauge we have

$$G^{00}(x-x') = -\frac{1}{4\pi}\,\frac{1}{|\mathbf{x}-\mathbf{x}'|}\,\delta(t-t')\,, \qquad G^{0i}(x-x') = G^{i0}(x-x') = 0\,,$$
$$G^{ij}(x-x') = -\left(\delta_{ij} - \partial_i\,\frac{1}{\nabla^2}\,\partial_j\right) G_0(x-x') =$$
$$= -\frac{1}{(4\pi)^4}\int d^4k\,\left(\delta_{ij} - \frac{k_h k_k}{\mathbf{k}^2}\right)\mathrm{P}\frac{1}{k^2}\,e^{-ik(x-x')} \tag{159}$$

and in a generic Lorentz gauge

$$G^{\mu\nu}_\lambda(x-x') = \frac{1}{(4\pi)^4}\int d^4k\,\left(g^{\mu\nu} - (1-\lambda)\,\mathrm{P}\frac{k_\mu k_\nu}{k^2}\right)\,\mathrm{P}\frac{1}{k^2}\,e^{-ik(x-x')}\,, \tag{160}$$

with the $\lambda$ specifying the specific gauge.

For consistency the classical source must be assumed to satisfy the continuity equation

$$\partial_\mu j^\mu(x) = 0 \tag{161}$$

and, in analogy with the cases of the harmonic oscillator and of the scalar field, we can avoid an explicit reference to the complicate lattice formulation

if we introduce the further restriction

$$\int d^4x\, d^4x'\, j_\mu(x) G^{\mu\nu}(x-x') j_\nu(x') = \int d^4\, \tilde{j}^*_\mu(k) \tilde{G}^{\mu\nu}(k) \tilde{j}_\nu(k) < 0\,. \qquad (162)$$

Under this assumption we can show that we can construct an operational density $\mathbf{f}(t_F,\, t_0;\, [f^{\mu\nu}])$, $f^{\mu\nu}(x) = \partial^\mu a^\nu(x) - \partial^\nu a^\mu(x)$ being the classical e.m. field and $a^\mu(x)$ the classical tetra-potential, and from this to derive a positive probability distribution on the space of the histories of the classical field. On the other side we shall also see that eq. (156) is consistent with local conservation of energy and momentum.

These results hold both for the free e.m. field and for spinor Electrodynamics, what is much more interesting. Even i this second case only the e.m. field is treated as continuously monitored and interpreted as a classical field, no reference is made to matter quantities. We shall treated separately the two cases.

## 6.1 Free field

The density of Lagrangian is

$$L(x) = -\frac{1}{4}\, F_{\mu\nu}(x)\, F^{\mu\nu}(x) = \frac{1}{2}\, (\mathbf{E}^2(x) - \mathbf{B}^2(x)) \qquad (163)$$

and the energy momentum tensor

$$\hat{T}^{\mu\nu}_{\mathrm{em}}(x) = \hat{F}^{\mu\rho} \hat{F}_\rho^{\ \nu} + \frac{1}{4} g^{\mu\nu} \hat{F}_{\rho\sigma} \hat{F}^{\rho\sigma}\,. \qquad (164)$$

Specifically

$$T^{00}_{\mathrm{em}}(x) = \frac{1}{2}\, (\mathbf{E}^2 + \hat{\mathbf{B}}^2)\,, \qquad T^{0i}_{\mathrm{em}}(x) = (\mathbf{E} \times \hat{\mathbf{B}})_i\,. \qquad (165)$$

In the *Coulomb gauge* we have the conditional equations

$$A_0(x) = 0\,, \qquad \nabla \cdot \mathbf{A}(x) = 0\,, \qquad \nabla \cdot \mathbf{E}(x) = 0\,. \qquad (166)$$

and the *Quantization rules*

$$\begin{aligned} &[\hat{A}^i(t,\mathbf{x}),\, \hat{A}^j(t,\mathbf{x}')] = 0\,, \qquad [\hat{E}^i(t,\mathbf{x}),\, \hat{E}^j(t,\mathbf{x}')] = 0 \\ &[\hat{E}^i(t,\mathbf{x}),\, \hat{A}^j(t,\mathbf{x}')] = i\, (\delta_{ij} - \partial_i \frac{1}{\nabla^2}\, \partial_j) \delta^3(\mathbf{x} - \mathbf{x}')\,. \end{aligned} \qquad (167)$$

From these we can derive the commutation relations for the field

$$
\begin{aligned}
&[\hat{E}^i(t,\mathbf{x}),\, \hat{E}^j(t,\mathbf{x}')] = 0\,, \qquad [\hat{B}^i(t,\mathbf{x}),\, \hat{B}^j(t,\mathbf{x}'] = 0 \\
&[\hat{E}^i(t,\mathbf{x}),\, \hat{B}^j(t,\mathbf{x}')] = \, -i\,\epsilon_{ijl}\partial_l\, \delta^3(\mathbf{x}-\mathbf{x}')\,,
\end{aligned} \tag{168}
$$

which are independent of the gauge.

From eq, (168) follows immediately that $\mathcal{K}\,(x,\, j_\rho(x))$ as given by eqs. (156,158) satisfies the consistency relation

$$
[\mathcal{K}\,(x,\, j_\rho(x))\,,\, \mathcal{K}\,(x',\, j_\rho(x'))] = 0 \tag{169}
$$

on a spacelike surface and also at equal time

$$
\mathcal{L}'(x')\, \hat{T}^{0\nu}_{\text{em}}(x) = 0\,. \tag{170}
$$

Consequently

$$
\partial_\mu \langle \hat{T}^{\mu\nu}_{\text{em}}(x)\rangle_{QM} = 0\,. \tag{171}
$$

According to standard methods for path integral in gauge field theories the CFO takes the form

$$
\begin{aligned}
&\langle \mathbf{A}_F, \sigma_F | \mathcal{G}(\sigma_F,\, \sigma_0;\, [j_\rho]) | \mathbf{A}'_F,\, \sigma_F\rangle = \text{const} \int \mathcal{D}_{\sigma_0}\mathbf{A}_0 \qquad (172) \\
&\int \mathcal{D}_{\sigma_0}\mathbf{A}'_0\, \langle \mathbf{A}_0,\, \sigma_0 | \hat{\rho}_0 | \mathbf{A}'_0,\, \sigma_0\rangle \int_{\mathbf{A}_0}^{\mathbf{A}_F} \mathcal{D}\mathbf{A}\;\, \delta[\nabla\cdot\mathbf{A}] \int_{\mathbf{A}'_0}^{\mathbf{A}'_F} \mathcal{D}\mathbf{A}'\;\, \delta[\nabla\cdot\mathbf{A}'] \\
&\exp \int_{\sigma_0}^{\sigma_F} d^4x \left\{ \frac{\gamma}{16}(F_{\mu\nu} - F'_{\mu\nu})(F^{\mu\nu} - F'^{\mu\nu})) + \frac{i}{4}(F_{\mu\nu}F^{\mu\nu} - F'_{\mu\nu}F'^{\mu\nu}) - \right. \\
&\left. -\frac{i}{2}\, j_\mu\, (A^\mu + A'^\mu) + \frac{1}{2\gamma}\int_{\sigma_0}^{\sigma_F} d^4x'\, j_\mu(x)\, G^{\mu\nu}(x-x')\, j_\nu(x') \right\}\,,
\end{aligned}
$$

where consistently with (161) we can assume

$$
j_0(x) = 0\,, \qquad\qquad \nabla\cdot\mathbf{J}(x) = 0 \tag{173}
$$

and then (159) becomes

$$
\int d^4k\, \tilde{j}^*(k) \left(\delta_{ij} - \frac{k_ik_j}{\mathbf{k}^2}\right) j_j(k)\, \mathrm{P}\,\frac{1}{k^2} = \int d^4k\, |\mathbf{j}(k)|^2 \mathrm{P}\,\frac{1}{k^2} > 0\,, \tag{174}
$$

which essentially implies that timelike $k$ prevail on spacelike ones.

From (172) we obtain by the usual procedure

$$\begin{aligned}
&\mathbf{f}(\sigma_F,\, \sigma_0;\, [a_\rho(x)])\, \hat{\rho} = &(175)\\
&= \int \mathcal{D}_c \mathbf{j}\, \delta[\nabla \cdot \mathbf{j}]\, \exp\left( i \int d^4x\, j_\mu(x)\, a^\mu(x) \right) \mathcal{G}(\sigma_F,\, \sigma_0;\, [j])\hat{\rho} = \\
&= C_\gamma\, \exp\left\{ \frac{\gamma}{4} \int d^4x\, [f_{\mu\nu}(x) - \hat{F}_{\mu\nu}(x)][f^{\mu\nu}(x) - \hat{F}^{\mu\nu}(x)] \right\} \hat{\rho} \\
&\qquad \exp\left\{ \frac{\gamma}{4} \int d^4x\, [f_{\mu\nu}(x) - \hat{F}_{\mu\nu}(x)][f^{\mu\nu}(x) - \hat{F}^{\mu\nu}(x)] \right\} ,
\end{aligned}$$

which is obviously positive.

## 6.2 Spinor Electrodynamics

Now let us consider the more significant model of spinor electrodynamics and assume as classical variables the e. m. field components alone as in the free case.

Any number of Dirac fields could be included in principle, however for simplicity we shall explicitly write only one field. We stress that we do not consider any classical variable relative to the Dirac field and any observation on the system is supposed to be expressed in terms of modifications on the classical e. m. field (see App. D an explicit discussion).

The Lagrangian density of the system is

$$\begin{aligned}
L(x) = -\frac{1}{4}\, F_{\mu\nu}\, F^{\mu\nu} - \frac{i}{2}(\bar{\psi}\gamma^\rho \partial_\rho \psi - \partial_\rho \bar{\psi}\gamma^\rho \psi) - \\
-m\, \bar{\psi}\psi - eA^\mu\, \bar{\psi}\gamma^\mu \psi\, . \qquad (176)
\end{aligned}$$

We shall use the same convention as in the preceding subsection and shall operate again in the Coulomb gauge. Then instead of (166) we have

$$\nabla \cdot \mathbf{A}(x) = 0\,, \qquad \text{but} \qquad \nabla \cdot \mathbf{E}(x) = e\bar{\psi}\gamma^0\psi \qquad (177)$$

and $\mathbf{E}(x) = \mathbf{E}_{\mathrm{T}}(x) + \mathbf{E}_{\mathrm{L}}(x)$ with

$$\begin{aligned}
\mathbf{E}_{\mathrm{T}}(x) = -\frac{\partial \mathbf{A}(x)}{\partial t}\,, \qquad \mathbf{E}_{\mathrm{L}}(x) = -\nabla A^0(x)\,, \\
A^0(t,\, \mathbf{x}) = -\frac{e}{4\pi} \int d^3\mathbf{y}\, \frac{1}{|\mathbf{x} - \mathbf{y}|}\, \bar{\psi}(t,\, \mathbf{y})\gamma^0\psi(t,\, \mathbf{y})\, . \qquad (178)
\end{aligned}$$

The basic commutation rules for the e. m. field are again given by eqs. (167,168) with the electric field replaced by its transverse part $E_h^{\mathbf{T}}(x)$. The remaining commutation rules are obviously

$$\begin{aligned}
&\{\hat{\psi}_\alpha(t,\,\mathbf{x}),\,\hat{\psi}_\beta(t,\,\mathbf{x}')\} = \{\bar{\hat{\psi}}_\alpha(t,\,\mathbf{x}),\,\bar{\hat{\psi}}_\beta(t,\,\mathbf{x}')\} = 0 \\
&\{\hat{\psi}_\alpha(t,\,\mathbf{x}),\,\bar{\hat{\psi}}_\beta(t,\,\mathbf{x}')\} = \gamma^0_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{x}')\,, \\
&[\hat{\psi}_\alpha(t,\,\mathbf{x}),\,\hat{A}^i(t,\,\mathbf{x}')] = [\bar{\hat{\psi}}_\alpha(t,\,\mathbf{x}),\,\hat{A}^i(t,\,\mathbf{x}')] = 0
\end{aligned} \tag{179}$$

and so

$$[\hat{\psi}_\alpha(t,\,\mathbf{x}),\,\hat{E}^i_{\mathbf{T}}(t,\,\mathbf{x}')] = [\bar{\hat{\psi}}_\alpha(t,\,\mathbf{x}),\,,\hat{E}^i_{\mathbf{T}}(t,\,\mathbf{x}')] = 0\,. \tag{180}$$

From the preceding equations there follow immediately

$$\begin{aligned}
&[\hat{\psi}_\alpha(t,\,\mathbf{x}),\,\hat{A}^0(t,\,\mathbf{x}')] = -\frac{e}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\,\psi_\alpha(t,\,\mathbf{x}) \\
&[\bar{\hat{\psi}}_\alpha(t,\,\mathbf{x}),\,\hat{A}^0(t,\,\mathbf{x}')] = \ \frac{e}{4\pi}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\,\bar{\hat{\psi}}_\alpha(t,\,\mathbf{x})
\end{aligned} \tag{181}$$

and

$$[\hat{A}^0(t,\,\mathbf{x}),\,\hat{A}^i(t,\,\mathbf{x}')] = 0\,, \qquad [\hat{A}^0(t,\,\mathbf{x}),\,\hat{E}^i_{\mathrm{T}}(t,\,\mathbf{x}')] = 0\,. \tag{182}$$

Then, (168) holds also for the total e. m. field, as should be by gauge invariance, $A^0(t,\,\mathbf{x})$ and consequently the longitudinal and the total electric fields commute with a bilinear expression of the form $\bar{\hat{\psi}}_\alpha(t,\,\mathbf{x}')\hat{\psi}_\beta(t,\,\mathbf{x}')$

$$[\hat{E}^i(t,\,\mathbf{x}),\,\bar{\hat{\psi}}_\alpha(t,\,\mathbf{x}')\hat{\psi}_\beta(t,\,\mathbf{x}')] = 0\,. \tag{183}$$

Since only the e. m. field is introduced as a classic variable $\mathcal{L}(x)$ and $\mathcal{K}(x,j)$ are again as defined by eqs. (156, 158) and the restriction on $j^\mu(x)$ is again expressed by eq. (162). The CFO and the density of operation remain of the form (172) and (175) but with the free e. m. Lagrangian replaced by the complete Lagrangian (176), the functional integral understood even on Clifford field $\psi$ and $\bar{\psi}$, evolution of the operators in the Heisenberg picture intended with respect to the total Hamiltonian $\hat{H} = \hat{H}_{\mathrm{em}} + \hat{H}_{\mathrm{D}} + \hat{H}_{\mathrm{int}}$.

Correspondingly the energy momentum tensor can be written

$$T^{\mu\nu}(x) = T^{\mu\nu}_{\mathrm{em}}(x) + T^{\mu\nu}_{\mathrm{D}}(x) + T^{\mu\nu}_{\mathrm{int}}(x)\,, \tag{184}$$

where $T^{\mu\nu}_{\rm em}(x)$ is again of the form (163),

$$T^{\mu\nu}_{\rm D}(x) = \frac{i}{2}(\bar{\psi}\gamma^{\mu}\partial^{\nu}\psi - \partial^{\nu}\bar{\psi}\gamma^{\mu}\psi) - \\ -g^{\mu\nu}[\frac{i}{2}(\bar{\psi}\gamma^{\rho}\partial_{\rho}\psi - \partial_{\rho}\bar{\psi}\gamma^{\rho}\psi) - m\bar{\psi}\psi] \qquad (185)$$

and

$$T^{\mu\nu}_{\rm int}(x) = g^{\mu\nu}e\bar{\psi}\gamma^{\rho}\psi A_{\rho}\,. \qquad (186)$$

Then from the commutation rules (167, 168, 179-183) one can immediately check that again

$$\mathcal{L}'(x')\,\hat{T}^{0\nu}(x) = 0\,. \qquad (187)$$

and so the local energy momentum conservation remains valid in the form

$$\partial_{\mu}\langle\hat{T}^{\mu\nu}(x)\rangle_{QM} = 0\,. \qquad (188)$$

also in this case. Similar conservation equation obviously are valid for the electric charge, the barionic number, etc.

Notice that the counterpart of (172) can be immediately rewritten in a generic Lorentz gauge as

$$\langle\mathbf{A}_F,\,\zeta_F,\,\sigma_F|\mathcal{G}(\sigma_F,\,\sigma_0;\,[j_{\rho}])|\mathbf{A}'_F,\,\zeta'_F,\,\sigma_F\rangle = \\ = \sum_{\zeta_0\zeta'_0}\int\mathcal{D}_{\sigma_0}\mathbf{A}_0\,\mathcal{D}_{\sigma_0}\mathbf{A}'_0\langle\mathbf{A}_0,\,\zeta_0,\,\sigma_0|\hat{\rho}_0|\mathbf{A}'_0,\,\zeta'_0,\,\sigma_0\rangle \\ \int_{\mathbf{A}_0}^{\mathbf{A}_F}\mathcal{D}A\;\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\int_{\mathbf{A}'_0}^{\mathbf{A}'_F}\mathcal{D}A'\;\mathcal{D}\bar{\psi}'\,\mathcal{D}\psi' \\ \exp\int_{\sigma_0}^{\sigma_F}d^4x\left\{\frac{\gamma}{8}(F_{\mu\nu} - F'_{\mu\nu})(F^{\mu\nu} - F'^{\mu\nu}))-\right. \\ -\frac{i}{2}\,j_{\mu}\,(A^{\mu} + A'^{\mu}) + \frac{1}{4\gamma}\int_{\sigma_0}^{\sigma_F}d^4x'\;j_{\mu}(x)\,G^{\mu\nu}(x - x')\,j_{\nu}(x')+ \\ \left.+i[L_{\rm eff}(A,\,\bar{\psi},\,\psi) - L_{\rm eff}(A',\,\bar{\psi}',\,\psi')]\right\}\;, \qquad (189)$$

where $\zeta_0$, $\zeta'_0$, $\zeta_F$, $\zeta'_F$ specify initial and final states of the spinor field,

$$L_{\rm eff}(A,\,\bar{\psi},\,\psi) = L(A,\,\bar{\psi},\,\psi) - \frac{1}{2\lambda}(\partial_{\mu}A^{\mu}) \qquad (190)$$

and $G^{\mu\nu}_{\lambda}(x - x')$ defined by eq. (160).

Finally let us consider the fluctuations of the classical e. m. field around its expectation value. Formally we can write

$$(-i)^2 \frac{\delta}{\delta j^\mu(x)} \frac{\delta}{\delta j^\nu(x')} \mathrm{Tr}\, [\mathcal{G}(\sigma_F,\, \sigma_0;\, [j_\rho])\hat{\rho}_0]|_{j=0} =$$

$$= \frac{1}{2\gamma}\, G_{\mu\nu}(x-x') + \theta(t-t')\, \frac{1}{2}\, \mathrm{Tr} \left[ \hat{A}_\mu(x) \mathcal{G}(\sigma,\, \sigma')\, \{\hat{A}_\nu(x'),\, \mathcal{G}(\sigma',\, \sigma_0)\hat{\rho}_0\} \right] +$$

$$+\theta(t'-t)\, \frac{1}{2}\, \mathrm{Tr} \left[ \hat{A}_\nu(x') \mathcal{G}(\sigma',\, \sigma)\, \{\hat{A}_\mu(x),\, \mathcal{G}(\sigma,\, \sigma_0)\hat{\rho}_0\} \right] \cong$$

$$\cong \frac{1}{2\gamma}\, G_{\mu\nu}(x-x') \,+\, \langle A_\mu(x)\, A_\nu(x') \rangle_{\mathrm{QM}}\,, \tag{191}$$

if $t'$ is sufficiently close to $t$, and

$$\langle f_{\mu\nu}(x)\, f_{\rho\sigma}(x') \rangle = \frac{1}{2\gamma} \left[ \partial_\mu \partial'_\rho\, G_{\nu\sigma}(x-x') - \partial_\mu \partial'_\sigma\, G_{\nu\rho}(x-x') - \right.$$

$$-\partial_\nu\, \partial_\rho G_{\mu\sigma}(x-x') + \partial_\nu\, \partial_\sigma\, G_{\mu\rho}(x-x')(x-x')] +$$

$$+ \langle \hat{F}_{\mu\nu}(x)\, \hat{F}_{\rho\sigma}(x') \rangle_{\mathrm{QM}}\,. \tag{192}$$

Then, let us set $E^i_{\mathrm{class}}(x) = f_{0i}(x)$, $B^i_{\mathrm{class}}(x) = \frac{1}{2}\epsilon_{ijl}\, f_{jl}(x)$ and

$$\mathbf{E}_h(x) = \int d^4x'\, h(x-x')\, \mathbf{E}_{\mathrm{class}}(x') \qquad \mathbf{B}_h(x) = \int d^4x'\, h(x-x')\, \mathbf{B}_{\mathrm{class}}(x')\,, \tag{193}$$

with $h(x-x')$ as in (155). Obviously we have

$$\langle \mathbf{E}_{\mathrm{class}}(x) \rangle = \langle \hat{\mathbf{E}}(x) \rangle_{\mathrm{QM}}\,, \qquad \langle \mathbf{B}_{\mathrm{class}}(x) \rangle = \langle \hat{\mathbf{B}}(x) \rangle_{\mathrm{QM}} \tag{194}$$

and, again for $\bar{k} \sim 2/\tau$,

$$\langle (E^i_h - \langle E^i_h \rangle)^2 \rangle = \frac{1}{2\gamma} \int d^4k\, \mathrm{P}\, \frac{k_0^2 - k_i^2}{k^2}\, |\tilde{h}(k)|^2 \,+\, \langle (\hat{E}^i_h - \langle \hat{E}^i_h \rangle)^2 \rangle_{\mathrm{QM}} \sim$$

$$\sim \frac{0.2}{\gamma\, \tau\, a^3} + \langle (\hat{E}^i_h - \langle \hat{E}^i_h \rangle)^2 \rangle_{\mathrm{QM}}\,, \tag{195}$$

A similar equation can be derived for $B^i_h$.

# 7 Conclusive consideration

In conclusion we have shown on three different models that it is possible modify the formalism for the continuous monitoring of *macroscopic quanti-*

*ties* in Quantum Theory in such a way that the basic conservation laws are preserved.

As we mentioned the idea is that Quantum Theory should be modified by introducing certain basic macroscopic quantities, that are formally treated as continuously monitored, but are actually thought as *classical quantities* or *beables*. These are supposed to have well determined values at each time and in terms of modifications of them any other observation should be expressed.

Obviously the most significant of the models we have proposed is the spinor electrodynamics, in which the macroscopic electromagnetic field components are considered as *classical*. Even this to be made realistic should be extended at least to the so called particle *Standard Model*. In the usual formulation of the latter, in which the Higgs is treated as elementary, this may be not a trivial task. The difficulty comes from the occurrence of terms quadratic in the e.m. potential in the boson sector of the theory. However there are some important properties of the model that should remain valid in a more complete theory; let us briefly discuss them.

First of all, note that it is implicitly built in the eqs. (23-27) (that remain valid at least as good approximations) that, when applied to a small number of particles, the formalism reproduces the usual quantum theory. Two essential modifications occur:

1) only observables that can be expressed in a modification of the above macroscopic field have to be considered;

2) the usual unitary evolution has to be corrected by the action of the mapping $\mathcal{G}(t,\, t_a)$ on the initial density operator $\hat{\rho}(t_a)$.

In the context, small number of particles means compatible with a negligible macroscopic e. m. field.

A detailed discussion of the question is given in App. D. Here we want rather comment the meaning of the two statements.

Point 1). This should raise no problems. In fact, practically all our particle detectors work in terms of e. m. effects, that by appropriate amplification reach the macroscopic scale. In last analysis, even in the spirit of von Neumann psycho-physical parallelism, the states of our brain related to our perceptions are expressed in terms of membrane potentials, action potentials, charge distributions and so on.

Point 2). The entity of the corrections to time evolution is controlled by the value of the constant $\gamma$, that has to be intended as a new fundamental constant of nature It is clear that, since ordinary quantum theory works well for few particles, $\gamma$ should be small. On the other side in eq. (194) $\gamma$ occurs in the denominator of the variance of the field. Now it is clear that, in order the all idea to make sense, such variance has to be negligible at some typical macroscopic scale. That is for some reasonable values of $\tau$ and $a$ in (194) and some appropriate $\mathbf{E}_{\rm typ}$ or $\mathbf{B}_{\rm typ}$ we must have

$$\langle(\mathbf{E}_h - \langle\mathbf{E}_h\rangle)^2\rangle/\mathbf{E}_{\rm typ}^2 \ll 1\,, \qquad \langle(\mathbf{B}_h - \langle\mathbf{B}_h\rangle)^2\rangle/\mathbf{B}_{\rm typ}^2 \ll 1\,. \tag{196}$$

This provide us the lower bound

$$\gamma \gg 0.2/\left(\mathbf{E}_{\rm typ}^2\,\tau\,a^3\right)\,. \tag{197}$$

and a similar for $\mathbf{B}_{\rm typ}$. To see what this means let us take $\mathbf{E}_{\rm typ}^2 \sim \mathbf{B}_{\rm typ}^2$ as the value of equilibrium inside a cavity at ordinary temperature $T = 300\,\mathrm{K}$ and e. g. $\tau = 1\,\mathrm{ms} = 3\times 10^7\,\mathrm{cm}$, $a = 1\,\mu\mathrm{m} = 10^{-4}\,\mathrm{cm}$. For the density e. m. energy the Stefan-Boltzman law gives in natural units

$$u(T) = 7.56\times 10^{-15}\,T^4\,\mathrm{erg\ cm}^{-1} \ = 2.39\times 10^2\,T^4\,\mathrm{cm}^{-4}\,. \tag{198}$$

Then, setting

$$\mathbf{E}_{\rm typ}^2 \sim \mathbf{B}_{\rm typ}^2 \sim u(300\ \mathrm{K}) = 1.44\times 10^{18}\,cm^{-4}\,, \tag{199}$$

eq. (197) becomes

$$\gamma \gg 5\times 10^{-15} \tag{200}$$

which should not raises problems too.

Second, to make a comparison with the collapse models, let us observe that from the mathematical point of view our proposal corresponds to a specific choice of the “dissipative” term in the Liouville-von Newman equation, dropping the positivity requirement at the price of restricting the class of the observables.The result is the possibility to reformulate the theory in such a way that the interference terms among macroscopic states do not decay but are conceptually suppressed.

Finally let us stress that an equation of the type (4) breaks temporal inversion invariance and this should have astrophysical and cosmological consequences. Possibly it is just by such a kind of consequences that a theory of

this type could be tested and the new fundamental constant $\gamma$ determined. More in general, for what concerns experimental tests see e. g. the discussion in [9] for the case of collapse models, that in part should apply even to the present one.

# 8 Acknowledgements

Warm thanks are due to my friend L. Lanz for many interesting discussion and critical remarks.

# 9 Appendices

# A Path integral for the harmonic oscillator amplitude

Reintroducing the mass explicitly the Lagrangian of the harmonic oscillator can be written

$$L = \frac{m}{2}(\dot{Q}^2 - \omega^2 Q^2) \tag{201}$$

and we have for the ordinary transition amplitude for an infinitesimal time integral

$$\langle Q',\, t+\epsilon | Q,\, t\rangle = \sqrt{\frac{m}{2\pi i \epsilon}} \exp\left\{ i\, \frac{m}{2} \left[ \frac{1}{\epsilon}\,(Q'-Q)^2 - \epsilon\, \omega^2\, Q^2 \right] \right\} . \tag{202}$$

Then, in our notation the path integral expression for a finite time interval is

$$\begin{aligned} \langle Q,\, t | Q_0,\, t_0\rangle = \left(\tfrac{m}{2\pi i \epsilon}\right)^{N/2} \int dQ_1\, dQ_2\, \ldots\, dQ_{N-!} \\ \exp\left\{ i\, \tfrac{m}{2} \textstyle\sum_{j=0}^{N-1} \left[ \tfrac{1}{\epsilon}\,(Q_{j+1}-Q_j)^2 - \epsilon\, \omega^2\, Q_j^2 \right] \right\} , \end{aligned} \tag{203}$$

the limit for large $N$ being obviously understood.

The above integral can be explicitly performed step by step and we obtain

$$\langle Q,\, t|Q_0,\, t_0\rangle = \sqrt{\frac{m\omega}{2\pi i \sin\omega\tau}}\; \exp\left\{ i m\,\omega\, \frac{(Q^2+Q_0^2)\, \cos\omega\tau - 2QQ_0}{2\sin\omega\tau} \right\}, \tag{204}$$

where $\tau = t - t_0$ [17].

To prove this is sufficient to observe that (204) reduces to (202) for an infinitesimal interval and it reproduce itself after a further infinitesimal step

$$\begin{aligned} \langle Q',\, t+\epsilon|Q_0,\, t_0\rangle &= \int dQ\, \langle Q',\, t+\epsilon|Q,\, t\rangle\langle Q,\, t|Q_0\, t_0\rangle = \\ &= \tfrac{m}{2\pi i}\sqrt{\tfrac{\omega}{\epsilon\sin\omega\tau}} \int dQ\, \exp\left\{ i\tfrac{m}{2}\left[\left(\tfrac{1}{\epsilon} - \epsilon\omega^2 + \omega\tfrac{\cos\omega\tau}{\sin\omega\tau}\right) Q^2 \right.\right. \\ &\qquad \left.\left. -2\left(\tfrac{1}{\epsilon}Q' + \tfrac{\omega}{\sin\omega\tau}\, Q_0\right) Q + i\tfrac{1}{\epsilon}Q'^2 + \omega\tfrac{\cos\omega\tau}{\sin\omega\tau}\, Q_0^2\right]\right\} = \\ &= \sqrt{\tfrac{m\omega}{2\pi i\, \sin\omega(t+\epsilon)}}\; \exp\left\{ im\omega\, \tfrac{(Q'^2+Q_0^2)\, \cos\omega(\tau+\epsilon) - 2Q'Q_0}{2\sin\omega(\tau+\epsilon)} \right\}, \end{aligned} \tag{205}$$

up to terms of order $\epsilon^2$ in the last equality.

For real $m$, actually the the above integral would be undetermined and it is usually made well defined by performing an infinitesimal rotation in the $t$ complex plane. On the contrary, note that, due to the prevalence of the term $\frac{1}{\epsilon}$ in the coefficient of $Q^2$ everything becomes perfectly defined if we put $m = i\mu$ with $\mu > 0$, independently of the sign of the other two terms. Then the situation becomes strictly similar to the one encountered in eqs (65) and (66).

# B Positivity of the basic matrix

Let us denote by $K_{ij}^{(n)}(\lambda)$ the $n\times n$ matrix

$$K_{ij}^{(n)}(\lambda) = \begin{pmatrix} 1-\lambda & -1 & 0 & \dots & 0 & 0 \\ -1 & 2-\lambda & -1 & \dots & 0 & 0 \\ 0 & -1 & 2-\lambda & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2-\lambda & -1 \\ 0 & & 0 & 0\dots & -1 & 1 \end{pmatrix}, \tag{206}$$

by $\bar{K}_{ij}^{(n)}(\lambda)$ the $n\times n$ obtained by suppressing the last row and column in $K_{ij}^{(n+1)}(\lambda)$ and by $\bar{\bar{K}}_{ij}^{(n)}(\lambda)$ the matrix obtained by suppressing the first row

and column in $\bar{K}_{ij}^{(n+1)}(\lambda)$. Obviously, with reference to the main text, we have $\bar{K}_{ij} = K_{ij}^{(N)}(\epsilon^2\omega^2)$.

Let us further denote by $D_n$, $\bar{D}_n$, $\bar{\bar{D}}_n$ the determinants of $K_{ij}^{(n)}(0)$, $\bar{K}_{ij}^{(n)}(0)$ and $\bar{\bar{K}}_{ij}^{(n)}(0)$ respectively. By developing them with respect to the first row, we obtain the recurrence relations

$$D_n = \bar{D}_{n-1} - \bar{D}_{n-2}\,, \quad \bar{D}_n = \bar{\bar{D}}_{n-1} - \bar{\bar{D}}_{n-2}\,, \quad \bar{\bar{D}}_n = 2\,\bar{\bar{D}}_{n-1} - \bar{\bar{D}}_{n-2}\,. \tag{207}$$

By induction from the last equation we have

$$\bar{\bar{D}}_n = n + 1 \tag{208}$$

and by using such result from the second and the first equation

$$\bar{D}_n = 1 \quad \text{and} \quad D_n = 0\,. \tag{209}$$

. Let us now consider the matrix $\bar{K}_{ij}^{(n)}(0)$ and note that this is positive, since all its principal subdeterminants are of the type $\bar{D}_p$ or $\bar{\bar{D}}_p$ with $p \leq n$ and so their are positive (we call principal determinants those obtained suppressing any number of rows and the corresponding columns in the original determinant). Then the eigenvalues of $\bar{K}_{ij}^{(n)}(0)$ are all positive and are given by the roots of the polynomial

$$\det \bar{K}_{ij}^{(n)}(\lambda) \equiv \det(\bar{K}_{ij}^{(n)}(0) - \lambda\delta_{ij}) = A_0 - A_1\lambda + A_2\lambda^2 - \ldots + (-\lambda)^{n-1}\,, \tag{210}$$

where the coefficients $A_0$, $A_1$, ... are expressed as sum of principal determinant and are again positive. In particular we have

$$A_0 = \bar{D}_n = 1, \qquad A_1 = \bar{\bar{D}}_{n-1} + (n-2)\bar{D}_{n-2} = 2(n-1). \tag{211}$$

Notice that, in order the matrix $\bar{K}_{ij}^{(n)}(\lambda)$ be itself positive, $\lambda$ must be smaller than the minimum eigenvalue $\lambda_{\mathrm{m}}$ of $\bar{K}_{ij}^{(n)}(0)$ and, since $A_2 > 0$, it must be $\lambda_m > A_0/A_1 = 1/2(n-1)$. Then, if

$$\omega^2\,\epsilon^2 \equiv \frac{\omega^2\,T^2}{N^2} < \frac{1}{2(N-1)} \quad \text{for} \quad N > 2\,\omega^2\,T^2\,, \tag{212}$$

$\bar{K}_{ij} \equiv \bar{K}_{ij}^{(N)}(\epsilon^2\omega^2)$ is certainly positive, as we stated.

## C Fourier transform of the interpolating continuous world line

Let us consider the continuous function defined in the entire interval $(t_0,\, t_F)$ by

$$Q(t) = Q_{j-1} + (t - t_{j-1})\frac{Q_j - Qj_1}{\epsilon} \qquad \text{for} \quad t \in (t_{j-1},\, t_j)\,, \tag{213}$$

for every $j = 0,\, 1,\, ...\, N$. This function interpolates the values $Q_j$ and makes the exponential in eq.(68) identical to those in eq.(66) up to to a terms of order of $\epsilon$ coming from the potential part of the action.

The Fourier coefficients of such function are given by

$$\begin{aligned}\tilde{Q}_k = \tfrac{1}{\sqrt{T}} \textstyle\int_{t_0}^{t_F} dt\, e^{-ikt}\, Q(t) = \sum_{j=0}^{N-1} e^{-ik\frac{t_{j+1}+t_j}{2}} \\ \left[\tfrac{Q_{j+1}+Q_j}{2}\, \epsilon\, \tfrac{\sin k\epsilon/2}{k\epsilon/2} + (Q_{j+1} - Q_j)\, \tfrac{i}{k} \left(\cos k\epsilon/2 - \tfrac{\sin k\epsilon/2}{k\epsilon/2}\right)\right]\,.\end{aligned} \tag{214}$$

The first term in the above expression is important for small $k$ but it is of the order of $\epsilon$, the second term is of the order of unity and it is important for $k\epsilon \sim \pi$ and so for $|k| \gg \omega$ if $\epsilon \ll \frac{2\pi}{\omega}$. Consequently the region $|k| < \omega$ gives a vanishing contribution for $\epsilon \to 0$ $(N \to \infty)$.

## D Ricovery of ordinary Quantum Mechanics for a small system

In the perspective of the paper any observation on a system has to be expressed in terms of the modification that the system induces on the *classical* e. m. field.

Let us consider, e. g., a system of a small number of particles characterized by a certain set of invariants (a total electric charge, baryon number, lepton number, etc) to which we shall refer as the object system. Let us assume that such particles interact freely among themselves during a certain interval of time $(t_a,\, t_b)$. We admit any kind of rearrangement inside the system, exchange of energy and momentum, production or destruction of particles, but no interaction with the external environment during such interval of time.

We assume that at the time $t_b$ the system comes in contact with an apparatus, by which the specific type of final particles, their momenta, energies etc. can be detected. To be specific we may think of the apparatus as a set of counters, filling densely a certain region kept under the action of a magnetic field.

Both the object system and the apparatus in their specific states must be thought as states of the same system of fields and can be expressed as appropriate composed creator operators applied to the vacuum state. Let us denote by $|u_1(t)\rangle$; $|u_2(t)\rangle$, ... and $|U_1(t)\rangle$; $|U_2(t)\rangle$, ... two orthogonal basis in the subspaces of the object system and of the apparatus and write

$$|u_j(t)\rangle = \hat{a}_j^\dagger(t)|0\rangle \qquad |U_r(t)\rangle = \hat{A}_r^\dagger(t)|0\rangle\,, \tag{215}$$

$\hat{a}_j^\dagger(t)$ and $\hat{A}_j^\dagger(t)$ being ordinary Heisenberg picture operators.

Then let us assume the object system alone to be described at the initial time $t_a$ by the statistical operator

$$\begin{aligned}\hat{\rho}^{\mathrm{O}}(t_a) = \sum_{ij} |u_i(t_a)\rangle \rho_{ij}(t_a)\langle u_j(t_a)| = \\ = \sum_{ij} \hat{a}^\dagger(t_a)|0\rangle \rho_{ij}(t_a)\langle 0|\hat{a}_i(t_a)\,.\end{aligned} \tag{216}$$

The assumption the system to be small implies that the classic e. m. field stays negligible in the region occupied by the system until this does not come in contact with the apparatus. So at the time $t_b$ we have

$$\begin{aligned}\hat{\rho}^{\mathrm{O}}(t_b) = \mathcal{G}(t_b,\, t_a)\{\sum_{ij} \hat{a}_i^\dagger(t_a)|0\rangle \rho_{ij}(t_a)\langle 0|\hat{a}_j(t_a)\} = \\ = \sum_{ij} \hat{a}_i^\dagger(t_b)|0\rangle \rho_{ij}(t_b)\langle 0|\hat{a}_j(t_a)\,,\end{aligned} \tag{217}$$

where

$$\rho_{ij}^{\mathrm{O}}(t_b) = \langle u_i(t_b)|\mathcal{G}(t_b,\, t_a)\{\sum_{kl} \hat{a}_k^\dagger(t_a)|0\rangle \rho_{kl}(t_a)\langle 0|\hat{a}_l(t_a)\}|u_j(t_b)\rangle\,. \tag{218}$$

Similarly let be

$$\hat{\rho}^{\mathrm{A}}(t_a) = \sum_{rs} \hat{A}_r^\dagger(t_a)|0\rangle \rho_{rs}(t_a)\langle 0|\hat{A}_s(t_a) \tag{219}$$

the initial state of the apparatus. In this case we can assume that the counters remain in their charged states corresponding to the classical e. m. field having certain specific stable values inside them until any interaction with some external object occurs. Again this corresponds to the classical world history of the electric field $\mathbf{E}_{\text{classic}}(t,\,\mathbf{x})$ falling with certainty in a set $M_0 \in \Sigma_{t_a}^{t_b}$, being null the probability of occurrence of the complementary set $M_0'$. Then

$$\mathcal{F}(M_0; t_b\, t_a)\hat{\rho}^{\mathrm{A}}(t_a) = \mathcal{G}(t_b,\, t_a)\hat{\rho}^{\mathrm{A}}(t_a) \tag{220}$$

and

$$\hat{\rho}^{\mathrm{A}}(t_b) = \sum_{rs} \hat{A}_r^{\dagger}(t_b)|0\rangle \rho_{rs}^{\mathrm{A}}(t_b)\langle 0|\hat{A}_s(t_a) \tag{221}$$

with again

$$\rho_{rs}^{\mathrm{A}}(t_b) = \langle U_r(t_b)|\mathcal{G}(t_b,\, t_a)\left\{\sum_{kl} \hat{A}_k^{\dagger}(t_a)|0\rangle \rho_{kl}^{\mathrm{A}}(t_a)\langle 0|\hat{A}_l(t_a)\right\}|U_s(t_b)\rangle\,. \tag{222}$$

Since we have assumed that the object and the apparatus do not come in contact before $t_b$, during the interval $(t_a,\, t_b)$ they must evolve independently and at the time $t_b$ for their compound state we have

$$\begin{aligned}\hat{\rho}^{\mathrm{T}}(t_b) &= \sum_{ij}\sum_{rs} \hat{a}_i^{\dagger}(t_b)\hat{A}_r^{\dagger}(t_b)|0\rangle \rho_{ij}^{\mathrm{O}}(t_b)\,\rho_{rs}^{\mathrm{A}}(t_b)\langle 0|\hat{A}_s(t_b)\hat{a}_j(t_b) = \\ &= \sum_{ij}\sum_{rs} |u_i,\, U_r;\, t_b\rangle \rho_{ij}^{\mathrm{O}}(t_b)\,\rho_{rs}^{\mathrm{A}}(t_b)\langle u_j,\, U_s; t_b|\,.\end{aligned} \tag{223}$$

In a subsequent time interval $(t_b,\, t_c)$, as consequence of the interaction with the particles of the object system some of the counter shall discharge and every specific pattern of discharged counters is interpreted as corresponding to certain specific particles with specific energies and momenta present in the system. Then, if we denote by $N \in \Sigma_{t_b}^{t_c}$ the set of classical e. m. world histories corresponding to the parameters specifying the particles types, energies, momenta etc. falling in a certain set $T$, we have

$$\begin{aligned}p(T,\, t_b) &= P(t_c,\, t_b;\, N) = \\ &\mathrm{Tr}[\mathcal{F}(t_c,\, t_b;\, N)\,\hat{\rho}^{\mathrm{T}}(t_b)] = \sum_{jj'} F_{j'j}(T,\, t_b)\,\rho_{jj'}^{\mathrm{O}}(t_b)\,,\end{aligned} \tag{224}$$

which is positive and can be confronted with (13) and where obviously

$$F_{j'j}(T,\,t_b) = \sum_{ir} \langle u_i(t_b),\,U_r(t_b)|\mathcal{F}(t_c,\,t_b;\,N)\left\{\sum_{ss'} |u_j(t_b),\,U_s(t_b)\rangle\right.$$
$$\left.\rho^{\mathrm{A}}_{ss'}(t_b)\langle u_j(t_b)',\,U'_s(t_b)|\right\}|u_i(t_b),\,U_r(t_b)\rangle\,. \qquad (225)$$

To be more explicit let us assume that the vectors $|u_j(t_b)\rangle$ already correspond to a specifications of the of the state of the particles at the time $t_b$ and $N_j \in \Sigma_{t_b}^{t_c}$ the corresponding pattern of discharge of the counters, we can write

$$\mathcal{F}(t_c,\,t_b;\,N_j)\,\hat{\rho}^{\mathrm{T}}(t_b) =$$
$$= \rho^{\mathrm{O}}_{jj}(t_b)\,\mathcal{G}(t_c,\,t_b)\{\sum_{rs} |u_j,\,U_r;\,t_b\rangle\,\rho^{\mathrm{A}}_{rs}(t_b)\langle u_j,\,U_s;t_b|\}\,. \qquad (226)$$

from which, since $\mathcal{G}(t_c,\,t_b)$ is trace-preserving, it follows

$$p_j(t_b) \equiv P(t_c,\,t_b;\,N_j) = \mathrm{Tr}\{\mathcal{F}(t_c,\,t_b;\,N_j)\,\hat{\rho}^{\mathrm{T}}(t_b)\} =$$
$$= \rho^{\mathrm{O}}_{jj}(t_b)\mathrm{Tr}\{\sum_{rs} |u_j,\,U_r;\,t_b\rangle\,\rho^{\mathrm{A}}_{rs}(t_b)\langle u_j,\,U_s;t_b|\} = \rho^{\mathrm{O}}_{jj}(t_b)\,, \qquad (227)$$

that is the prescription of usual elementary Quantum Theory up to the correction introduced in (219) by the action of the mapping $\mathcal{G}(t_b,\,t_a)$.

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